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Algebraic Automata Theory

Sheet 1, 2017-11-09

Exercise 1 [10 Points]

Analyse the following polarities in ***rel*** when $B = 1$.

$$\begin{tikzcd}[A,B] \arrow[rr, bend right, "- \triangleright T"'] \arrow[rr, phantom, "\top"] & & {[B,C]^{\mathrm{op}}} \arrow[ll, bend right, "T \triangleleft -"']\end{tikzcd} \quad \text{as well as} \quad \begin{tikzcd}[C,B] \arrow[rr, bend right, "- \triangleright T^{\mathrm{op}}"'] \arrow[rr, phantom, "\top"] & & {[B,A]^{\mathrm{op}}} \arrow[ll, bend right, "T^{\mathrm{op}} \triangleleft -"']\end{tikzcd}$$

Turn them into adjunctions between the power-sets $A\mathbb{P}$ and $(C\mathbb{P})^{\mathrm{op}}$. What happens, if the relation $A \xrightarrow{T} C$ is left-adjoint, *i.e.*, a function $A \xrightarrow{h} C$?

Exercise 2 [12 Points]

Show in detail that the two characterizations of adjunctions $F \dashv G$ in ***Cat*** are equivalent:

(a) There exist natural transformations $\mathscr{A} \xRightarrow{\eta} FG$ and $GF \xRightarrow{\epsilon} \mathscr{B}$ subject to

$$\begin{tikzcd} & \mathscr{B} \arrow[rr, "\mathscr{B}"] \arrow[dr, "G"] & \arrow[d, phantom, "\Uparrow \epsilon"] & \mathscr{B} \\ \mathscr{A} \arrow[ur, "F"] \arrow[rr, "\mathscr{A}"'] & \arrow[u, phantom, "\Uparrow \eta"] & \mathscr{A} \arrow[ur, "F"'] & \end{tikzcd} \quad = \quad \begin{tikzcd} \mathscr{A} \arrow[r, bend left, "F"] \arrow[r, bend right, "F"'] \arrow[r, phantom, "\Uparrow id"] & \mathscr{B} \end{tikzcd}$$

$$\begin{tikzcd} \mathscr{B} \arrow[rr, "\mathscr{B}"] \arrow[dr, "G"'] & \arrow[d, phantom, "\Uparrow \epsilon"] & \mathscr{B} \arrow[dr, "G"] & \\ & \mathscr{A} \arrow[ur, "F"] \arrow[rr, "\mathscr{A}"'] & \arrow[u, phantom, "\Uparrow \eta"] & \mathscr{A} \end{tikzcd} \quad = \quad \begin{tikzcd} \mathscr{B} \arrow[r, bend left, "G"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Uparrow id"] & \mathscr{A} \end{tikzcd}$$

(b) There exitst a natural isomorphism

$$\begin{tikzcd} \mathscr{A}^{\mathrm{op}} \times \mathscr{B} \arrow[r, bend left, "{[-F,-]}"] \arrow[r, bend right, "{[-,-G]}"'] \arrow[r, phantom, "\Downarrow \iota"] & \boldsymbol{set} \end{tikzcd}$$

Exercise 3 [15 Points]

Australian "mate calculus": consider a 2-category $\mathcal{B}$, where the 1-cells are denoted by arrows of the form $\relbar\!\!\!\rhd$, while the left-adjoint 1-cells or *maps* are denoted by arrows of the form $\longrightarrow$.

Prove that there is a bijective correspondence between 2-cells of the following types:

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "R"'] & B \arrow[d, "S"] \\
C \arrow[r, "g"'] \arrow[ur, phantom, "\alpha"] & D
\end{tikzcd}
\qquad \text{and} \qquad
\begin{tikzcd}
A \arrow[d, "R"'] \arrow[dr, phantom, "\alpha'"] & B \arrow[l, "f'"'] \arrow[d, "S"] \\
C & D \arrow[l, "g'"]
\end{tikzcd}
$$

where $f \dashv f'$ and $g \dashv g'$.