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Advanced Automata Theory
Exercise Sheet 5

TU Kaiserslautern
Summer term 2016

Out: May 18, **Updated II, May 19:** Added Ex. 2 c), Fixed bug in Ex. 4 Due: May 23, 12:00

Exercise 1: Quantifier Elimination

a) Eliminate the quantifiers of the following formula using the method described in class:

$$\neg\forall x\colon 3x < 2y \vee y < 2x\,.$$

b) An existential Presburger formula is a formula as described by the following EBNF:

$$\varphi ::= t_1 < t_2 \mid t_1 = t_2 \mid \exists x\colon \varphi_1 \mid \varphi_1 \vee \varphi_2 \mid \varphi_1 \wedge \varphi_2\,.$$

In particular, it does not contain negations and it only contains the atomic predicates $<$ and $=$.

Prove that existential Presburger formulas are equally expressive as Presburger formulas.

Hint: Use Presburger's theorem.

Exercise 2: Parikh Images of Regular Languages

a) Prove that if $\mathcal{L} \in \mathrm{REG}_\Sigma$ is regular, $\Psi(\mathcal{L})$ is semilinear.

b) Prove that for each semilinear set $S \subseteq \mathbb{N}^d$, there is a regular language $\mathcal{L}$ over $\Sigma = \{a_1, ..., a_d\}$ with $S = \Psi(\mathcal{L})$.

c) Let $lsbf^{-1} : (\mathbb{B}^d)^* \to \mathbb{N}^d$ be the map that takes a word of binary vectors w and returns the vector of natural numbers $\vec{n}$ such that w is the "least significant bit first"-encoding of $\vec{n}$.

Present a regular language $\mathcal{L}$ over $\mathbb{B}^d$ such that

$$\left\{ lsbf^{-1}(w) \,\middle|\, w \in \mathcal{L} \right\} \subseteq \mathbb{N}^d$$

is not semilinear.

Exercise 3: Parikh Images of Context Free Languages

Use Parkihk's theorem to compute a representation for the semilinear set $\Psi\left(\mathcal{L}(G)\right)$ for the grammar G which has the rules:

a) $S \to ab \mid XZ,\ Z \to SY, X \to a,\ Y \to b$

b) $S \to XY \mid \varepsilon,\ X \to aSb,\ Y \to bSc$

Exercise 4: Closure Properties of Semilinear Sets

a) Let $S = \bigcup_{i \in \{1,\ldots,k\}} \mathcal{L}(c_i, P_i) \subseteq \mathbb{N}^d$ be semilinear. Prove that semilinear sets are closed under Kleene iteration:

$$\left\{ v_1 + \ldots + v_t \;\middle|\; t \in \mathbb{N} \text{ and } v_1, \ldots, v_t \in S \right\} = \bigcup_{J \subseteq \{1,\ldots,k\}} \mathcal{L}\Big(\sum_{i \in J} c_i, \bigcup_{i \in J} P_i \cup \{c_i\} \Big) .$$

b) To prove that semilinear sets are closed under intersection, we showed that the intersection of two linear sets $\mathcal{L}\big(c, \{u_1, \ldots, u_m\}\big)$ and $\mathcal{L}\big(d, \{v_1, ..., v_n\}\big) \subseteq \mathbb{N}^d$ is semilinear. We defined

$$B = \left\{ (x_1, \ldots, x_m, y_1, \ldots, y_n) \in \mathbb{N}^{m+n} \;\middle|\; \sum_{i=1}^{m} x_i u_i = \sum_{i=1}^{n} y_i v_i \right\} .$$

Let s_B be the set of minimal elements of $B \setminus \{0\}$ with respect to the product order $\leq^d$, i.e. $u \leq^d v$ if $u_c \leq v_c$ for all $c \in \{1, \ldots, d\}$.

Prove that $B = \mathcal{L}\big(0, s_B\big)$.

Hint: You may use that $\leq^d$ is well-founded.