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Exercises to the lecture
Algorithmic Automata Theory
Sheet 5

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Peter Chini Delivery until 29.05.2018 at 12:00

Exercise 5.1 (Aperiodic Monoids)
Let $(M, \cdot, 1_M)$ be a finite monoid. Recall that a subgroup G in M is a subset $G \subseteq M$ such that G is non-empty, $(G, \cdot, 1_G)$ is a monoid, and any element $g \in G$ has an inverse $g^{-1} \in G$: $g \cdot g^{-1} = 1_G$. Note that 1_G and 1_M might be different elements.

Prove that each subgroup of M is trivial, i.e. consists of exactly one element, if and only if M is an aperiodic monoid.

Hint: To show that M is aperiodic, consider the set $(a^{N+\ell})_{\ell \in \mathbb{N}} \subseteq M$, for $a \in M$ and N its idempotent power. Show that it is a subgroup.

Exercise 5.2 (Solution Space)
State a Presburger formula φ such that every bound variable (non-free variable) occurs in *precisely one* term and such that

$$Sol(\varphi) = \left\{ \begin{pmatrix} 2n+1 \\ n+3 \end{pmatrix} \middle| \; n \in \mathbb{N} \right\} \cup \left\{ \begin{pmatrix} 3n+1 \\ 2n+2 \end{pmatrix} \middle| \; n \in \mathbb{N} \right\}.$$

Exercise 5.3 (Parikh Image)
The *Parikh image* is a map $\Psi : \Sigma^* \to \mathbb{N}^{|\Sigma|}$. It sends each word w to the vector $\Psi(w)$, where the components store the number of occurrences of each letter in w. For a language $L \subseteq \Sigma^*$, let $\Psi(L) = \{\Psi(w) | w \in L\}$. As an example, consider the alphabet $\Sigma = \{a, b, c\}$. Then we have:

$$\Psi(ababcb) = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \text{ and } \Psi((aa)^*(bbb)^*) = \left\{ \begin{pmatrix} 2n \\ 3m \\ 0 \end{pmatrix} \middle| \; n, m \in \mathbb{N} \right\}.$$

Give an NFA A so that $\Psi(L(A)) = Sol(\varphi)$ for formula φ from Exercise 5.2.

Exercise 5.4 (Presburger to NFA)

a) Prove the correctness of the construction given in class: For every $q \in \mathbb{Z}$ and $w \in (\mathbb{B}^n)^*$, the automaton accepts w, starting from q if and only if w encodes $\vec{c}$ with $\vec{a} \cdot \vec{c} \leq q$.

b) Construct a finite automaton over $\mathbb{B}$ for the Presburger formula $x - 3y \leq 1$.

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