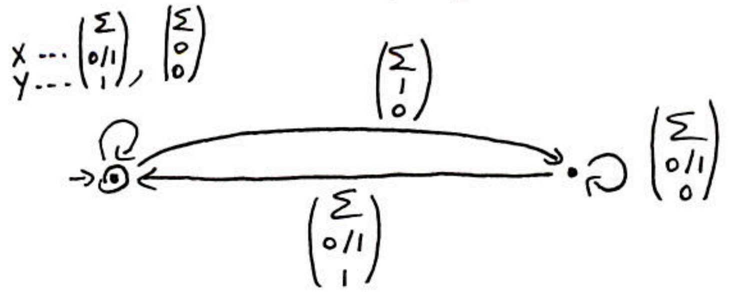
3.1. We must add another base case to the constructive proof of Büchi II, namely one that describes $A_{X \sqsubseteq Y}$:

$A_{X \sqsubseteq Y}$ over $\Sigma_{\{X,Y\}}$



3.2 „⇐" Let $L := L(\psi)$ for some $\forall$WMSO ψ. Since $\forall$WMSO $\subseteq$ WMSO, Büchi implies that, for any $\forall$WMSO φ there exists A_φ with $L(A_\varphi) = L(\varphi)$.

In particular, there exists A_ψ with $L = L(A_\psi)$ so L is regular.

„⇒" Let L be a regular language. Then $\overline{L}$ is also regular and, by Büchi (and its corrolary), there exists $\exists$WMSO $\psi := \exists X_1 : \ldots \exists X_n : \varphi$ such that $\overline{L} = L(\psi)$ (*).

Since $\overline{L(\psi)} = L(\neg\psi)$ for any WMSO ψ, from (*) we obtain that $L = L(\neg\psi)$ and since $\neg\psi = \forall X_1 : \ldots \forall X_n : \neg\varphi$ is $\forall$WMSO it means L is $\forall$WMSO-definable.

3.3 (a) We prove $G_k(uv^i w, uv^j w)$ with $i, j \geq 2^k - 1$ has a winning strategy for Duplicator.

IA. Duplicator wins $G_1(uvw, uv^n w)$ with $n \geq 1$ no matter what the Spoiler picks since it can match the letter in the other word.

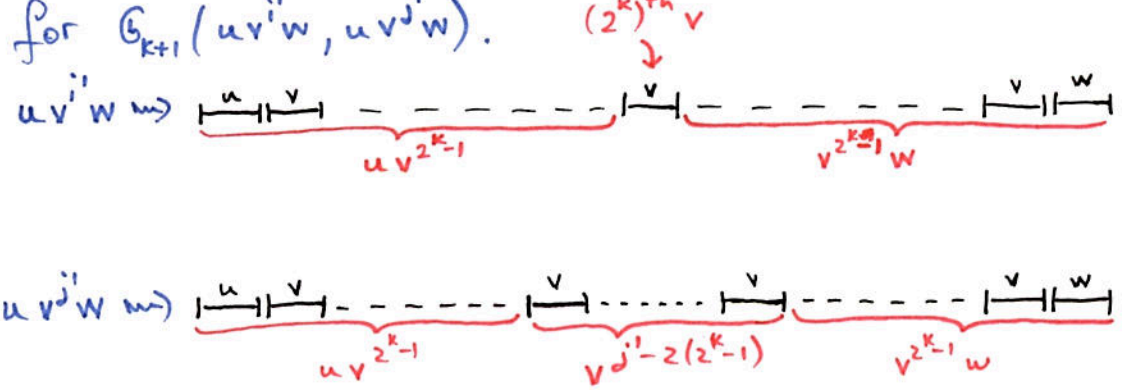
IS. Assume IH: Duplicator has a winning strategy for

$G_m(uv^i w, uv^j w)$ with $i, j \geq 2^k - 1$ and $m \leq k$.

Consider $G_{k+1}(uv^{i'} w, uv^{j'} w)$ with $i' = 2^{k+1} - 1 = 2 \cdot 2^k - 1$

$j' \geq 2^{k+1} - 1$.

We will argument why the Duplicator has a winning stretegy for $G_{k+1}(uv^{i'} w, uv^{j'} w)$.



We distinguish the following cases:

- Spoiler picks „s" in heading „uv^{2^k-1}"

  Then there is a one-to-one correspondence between uv^{2^k-1} in the two words which gives the choice of t by the Duplicator.

  Then $uv^{i'}w = u''\underline{t}u'v^{i}w$

  $uv^{j'}w = u''\underline{s}u'v^{j}w$ for some $j \geq i \geq 2^k-1$.

  Therefore, by $\underline{IH}$, $G_n(u'v^iw, u'v^jw)$ is won by Duplicator and through a one-to-one correspondence $G_m(u'', u'')$ for $m = k-n$ is also won by Duplicator.

  This proves why the Duplicator has a winning strategy in this case.

- Spoiler picks „s" in trailing „$v^{2^k-1}w$"

  This case is <u>really</u> similar to the previous one. In fact, it is symmetric to the previous case wrt. the given words.

- Spoiler picks „s" in middle „$v^{j'-2(2^k-1)}$"

  Then $uv^{j'}w = uv^x\underbrace{w'\underline{s}u'}_{=v}v^yw$ with $x, y \geq 2^k-1$. Let then the Duplicator choose matching $\underline{t}$ in the $(2^k)^{th}$ v of $uv^{i'}w$, i.e. $uv^{i'}w = uv^{2^k-1}\underbrace{w'\underline{t}u'}_{=v}v^{2^k-1}w$.

  By $\underline{IH}$, Duplicator has winning strategies both for $G_n(uv^{2^k-1}w', uv^xw')$ and $G_m(u'v^{2^k-1}w, u'v^yw)$ when $n, m \leq k$, therefore it has a winning strategy for the two games also when $n+m = k$.

  This proves why the Duplicator has a winning strategy even in this case.

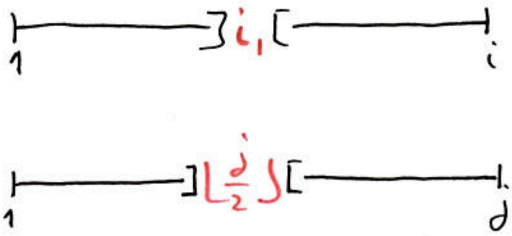
All in all, we can now conclude that the induction is complete and Duplicator has a winning strategy for

$$G_k(uv^iw, uv^jw) \text{ with } i, j \geq 2^k-1.$$

3.3. (b) Wlog. assume $i < j \leq 2^k - 1$. We describe a winning strategy for Spoiler when playing $G_k(a^i, a^j)$ as follows.

Let the Spoiler start by choosing position $\lfloor \frac{j}{2} \rfloor$ in a^j and assume the Duplicator matches it by some i_1 in $\{1, .., i\}$.

This separates both a^i and a^j in two (maybe empty) subwords. Since $i < j$, at least one of the left or right subword in a^i must be strictly smaller then the corresponding subword in a^j.



Following the above principle one will ultimately end up after a number n of rounds with

$$a < a^x, \quad x \geq 2$$

and the Duplicatore can then play only one more round.

If $a^{i-i_1} < a^{j - \underbrace{\lfloor \frac{j}{2} \rfloor}_{j_1}}$ look for "middle" of a^{j-j_1} and take this position as j_2, and so on...

So in at least $n+2$ rounds the Simulator wins by using the above strategy.

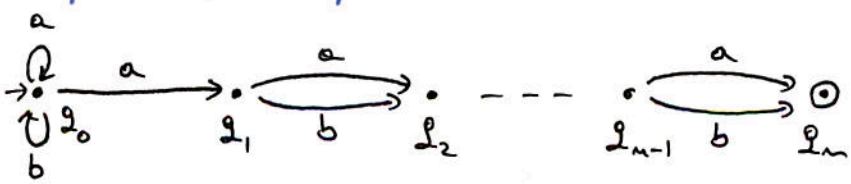
Since the strategy uses a halving principle and $j < 2^k - 1$, in n rounds the Duplicator vs. Simulator intervals will be of sizes $i_n < j_n < 2^{k-n} - 1$ so after k rounds the Duplicator cannot reply to the Simulator's pick.

As an example consider the following $G_3(a^6, a^7)$ game

$$\begin{array}{ccccccc} a & a & a & a & a & a & \\ & & & \top^1 & & \top^2 & \\ a & a & a & \underset{1}{a} & a & \underset{2}{a} & \underset{3}{a} \end{array}$$

The choices of the Simulator are in red and those of the Duplicator in black.

3.4. For the first part of the problem consider NFA A_n below:



with $q_{i-1} \xrightarrow{a} q_i$ and $q_{i-1} \xrightarrow{b} q_i$ for any $i \in \{2, \dots, n\}$. By construction it should be clear why $L_n = L(A_n)$.

For the second part, <u>assume</u> there is a DFA A'_n with <u>$< 2^n$ states</u> which accepts L_n. Using the <u>hint</u>, one should know there are precisely 2^n distinct words of length n over $\Sigma = \{a, b\}$.

Then $\left.\begin{array}{l} 2^n \text{ words} \\ < 2^n \text{ states} \end{array}\right\} \Rightarrow$ there exist two distinct "length n words" (say $\underline{uav}$ and $\underline{ubw}$) which starting from A'_n's initial state q'_0 lead to a same state q.

In other words $q'_0 \xrightarrow{uav} q$ and $q'_0 \xrightarrow{ubw} q$ for some q in A'_n. Therefore,

$$q'_0 \xrightarrow{uav} q \xrightarrow{u} q' \in Q_F \qquad (*)$$
$$q'_0 \xrightarrow{ubv} q \xrightarrow{u} q' \notin Q_F \qquad (**)$$

by extending the two words by $\underline{u}$.

Since $(*)$ and $(**)$ cannot both be true, we reached a contradiction to our assumption ∎