5.1. (a) The formula $u-1<3x \wedge 2x<t+6 \wedge x \equiv_2 1$ is already in DNF.
By uniformizing and eliminating the coefficients on x we get:

$$\exists x .\ 2u-2<x \ \wedge\ x<3t+18 \ \wedge\ x \equiv_{12} 6 \ \wedge\ x \equiv_6 0$$

Then $M = \mathrm{lcm}(12,6)=12$ and by eliminating x we obtain

$$\begin{aligned}
&\bigvee_{\rho=1}^{12} \big[-1 < -1+\rho \ \wedge\ 2u-2 < -1+\rho \\
&\qquad\qquad \wedge\ -1+\rho < 3t+18 \\
&\qquad\qquad \wedge\ -1+\rho \equiv_{12} 6 \ \wedge\ -1+\rho \equiv_6 0\big] \\
\vee &\bigvee_{\rho=1}^{12} \big[-1 < 2u-2+\rho \ \wedge\ 2u-2 < 2u-2+\rho \\
&\qquad\qquad \wedge\ 2u-2+\rho < 3t+18 \\
&\qquad\qquad \wedge\ 2u-2+\rho \equiv_{12} 6 \ \wedge\ 2u-2+\rho \equiv_6 0\big]
\end{aligned}$$

(b) The DNF of the formula under $\exists x$ is

$$\underbrace{(3x=1-t \wedge 1+u=2x)}_{\varphi_{==}} \vee \underbrace{(3x=1-t \wedge 1+u<2x)}_{\varphi_{=<}} \vee \underbrace{(3x<1-t \wedge 1+u=2x)}_{\varphi_{<=}}$$
$$\vee \underbrace{(3x<1-t \wedge 1+u<2x)}_{\varphi_{<<}} \vee \bigvee_{r\in\{0,1,3,4\}} \underbrace{x<0-u \wedge x\equiv_5 r}_{\varphi_r}$$

so after uniformizing and eliminating x in each of the sub-formulas we have:

$\exists x . \varphi_{==}$: $\ 2-2t = 3+3u \ \wedge\ 2-2t \equiv_6 0$

$\exists x . \varphi_{=<}$: $\ 3+3u < 2-2t \ \wedge\ 2-2t \equiv_6 0$

$\exists x . \varphi_{<=}$: $\ 3+3u < 2-2t \ \wedge\ 3+3u \equiv_6 0$

$\exists x . \varphi_{<<}$:
$$\begin{aligned}
&\bigvee_{\rho=1}^{6} \big[-1 < -1+\rho \ \wedge\ 3+3u < -1+\rho \\
&\qquad\qquad \wedge\ -1+\rho < 2-2t \\
&\qquad\qquad \wedge\ -1+\rho \equiv_6 0\big] \\
\vee &\bigvee_{\rho=1}^{6} \big[-1 < 3+3u+\rho \ \wedge\ 3+3u < 3+3u+\rho \\
&\qquad\qquad \wedge\ 3+3u+\rho < 2-2t \\
&\qquad\qquad \wedge\ 3+3u+\rho \equiv_6 0\big]
\end{aligned}$$

$\exists x . \varphi_r$: $\displaystyle \bigvee_{\rho=1}^{5} -1 < -1+\rho \ \wedge\ -1+\rho < 0-u \ \wedge\ -1+\rho \equiv_5 r$

5.2. (a) Let $S = \bigcup_{i=1}^{n} \mathcal{L}(c_i, P_i)$. One can decide $v \in S$ by deciding each of $v \in \mathcal{L}(c_i, P_i)$.

For a linear set $\mathcal{L}(c, P)$, it is decidable whether $v \in \mathcal{L}(c, P)$ since the set $\{v' \in \mathcal{L}(c,P) \mid \|v'\| \leq \|v\|\}$ is finite.

(b) Notice that $\mathcal{L}(c_1, P_1) + \mathcal{L}(c_2, P_2) = \mathcal{L}(c_1 + c_2, P_1 \cup P_2)$ for any two linear sets $\mathcal{L}(c_1, P_1)$, $\mathcal{L}(c_2, P_2)$. This is the case because $(c_1 + \sum_{i=1}^{n} k_{1i} \cdot p_{1i}) + (c_2 + \sum_{j=1}^{m} k_{2j} \cdot p_{2j}) = (c_1 + c_2) + (\sum_{i=1}^{n} k_{1i} \cdot p_{1i} + \sum_{j=1}^{m} k_{2j} \cdot p_{2j})$ for any $n, m \in \mathbb{N}$, $k_{1i}, k_{2j} \in \mathbb{N}$, and $p_{1i} \in P_1$, $p_{2j} \in P_2$.

Let $S_1 = \bigcup_{i=1}^{n} \mathcal{L}(c_{1i}, P_{1i})$ and $S_2 = \bigcup_{j=1}^{m} \mathcal{L}(c_{2j}, P_{2j})$ for some $n, m \in \mathbb{N}$. We will prove that $S_1 + S_2 = \bigcup_{i=1}^{n} \bigcup_{j=1}^{m} \mathcal{L}(c_{1i} + c_{2j}, P_{1i} \cup P_{2j})$ by double inclusion.

„$\subseteq$" Let $x \in S_1$, $y \in S_2$ such that $x + y \in S_1 + S_2$. Then $x \in \mathcal{L}(c_{1i}, P_{1i})$ and $y \in \mathcal{L}(c_{2j}, P_{2j})$ for some i, j within the appropriate limits.

This implies $x + y \in \mathcal{L}(c_{1i} + c_{2j}, P_{1i} \cup P_{2j}) \subseteq \bigcup_{i=1}^{n} \bigcup_{j=1}^{m} \mathcal{L}(c_{1i} + c_{2j}, P_{1i} \cup P_{2j})$.

„$\supseteq$" Let $v \in \bigcup_{i=1}^{n} \bigcup_{j=1}^{m} \mathcal{L}(c_{1i} + c_{2j}, P_{1i} \cup P_{2j})$. Then $v \in \mathcal{L}(c_{1i} + c_{2j}, P_{1i} \cup P_{2j})$ for some i, j within the appropriate limits.

Then $v \in \mathcal{L}(c_{1i}, P_{1i}) + \mathcal{L}(c_{2j}, P_{2j})$ and, since $\mathcal{L}(c_{1i}, P_{1i}) \subseteq S_1$, respectively $\mathcal{L}(c_{2j}, P_{2j}) \subseteq S_2$, it means that $v \in S_1 + S_2$.

5.3. (a) We prove $\psi(L)$ is semilinear by induction on $L \in REG_\Sigma$:

IA. $\psi(\emptyset) = \emptyset$ and $\psi(\{a\}) = \mathcal{L}\left(\begin{pmatrix} \vdots \\ 1 \\ \vdots \end{pmatrix}, \emptyset\right)$ are linear.

„a" entry

IS. Let $L, L_1, L_2 \in REG_\Sigma$ s.t. $\psi(L), \psi(L_1), \psi(L_2)$ are semilinear. We must consider the following cases:

(i)
$$\begin{aligned} \psi(L_1 \cup L_2) &= \{\psi(w) \mid w \in L_1 \cup L_2\} \\ &= \{\psi(w) \mid w \in L_1\} \cup \{\psi(w) \mid w \in L_2\} \\ &= \psi(L_1) \cup \psi(L_2). \end{aligned}$$

(ii)
$$\begin{aligned} \psi(L_1 . L_2) &= \{\psi(w_1 . w_2) \mid w_1 \in L_1 \text{ and } w_2 \in L_2\} \\ &= \{\psi(w_1) + \psi(w_2) \mid w_1 \in L_1 \text{ and } w_2 \in L_2\} \\ &= \psi(L_1) + \psi(L_2) \end{aligned}$$

(iii)
$$\begin{aligned} \psi(L^*) &= \{\psi(w_1 \ldots w_k) \mid k \in \mathbb{N} \text{ and } w_1, \ldots, w_k \in L\} \\ &= \{\psi(w_1) + \ldots + \psi(w_k) \mid k \in \mathbb{N}, w_i \in L\} = \psi(L)^* \end{aligned}$$

In case (i), $\psi(L_1)$ and $\psi(L_2)$ are finite unions of linear sets by IH, so $\psi(L_1) \cup \psi(L_2)$ is also a finite union of linear sets.

In case (ii), $\psi(L_1)$ and $\psi(L_2)$ are semilinear by IH, and by 5.2.b, $\psi(L_1) + \psi(L_2)$ is also semilinear.

In case (iii), $\psi(L) = \bigcup_{i=1}^{m} \mathcal{L}(c_i, P_i)$ by IH, so

$$\psi(L)^* = \bigcup_{I \subseteq \{1,\dots,m\}} \mathcal{L}\left(\sum_{i \in I} c_i, \bigcup_{i \in I} P_i \cup \{c_i\}\right).$$

This concludes the induction proof.

5.3. (b) Wlog. let $\Sigma = \{a_1, \dots, a_m\}$. Then for any linear set $\mathcal{L}(c, P)$ and for any $v \in P \cup \{c\}$ let $w_v := \underbrace{a_1 \dots a_1}_{v(1) \text{ times}} \dots \underbrace{a_i \dots a_i}_{v(i) \text{ times}} \dots \underbrace{a_m \dots a_m}_{v(m) \text{ times}}$.

For any $\mathcal{L}(c, P) \subseteq \mathbb{N}^m$ define $L_{c,P} = L\left(w_c + \left(\sum_{p \in P} w_p\right)^*\right)$ and if $S = \bigcup_{i=1}^{m} \mathcal{L}(c_i, P_i)$ define $L_S = \bigcup_{i=1}^{m} L_{c_i, P_i}$.

The fact that $\mathcal{L}(c, P) = \psi(L_{c,P})$ and $S = \psi(L_S)$ can then be seen as a direct implication of equalities (i)–(iii) in 5.3.(a) and the way $L_{c,P}$, L_S are defined.

5.4. (a) $L(r, S) = \emptyset$ iff. $\nexists w \in L(r)$ such that $\psi(w) \in S$

iff. $\psi(L(r)) \cap S = \emptyset$.

Now, either we argue that $\psi(L(r)) \cap S$ is semilinear by 5.3.a and closure under intersection of semilinear sets, in which case $L(r, S) = \emptyset$ is decidable since set emptiness is decidable over $\mathbb{N}^m$, or we argue that there exists $L \in REG_\Sigma$ such that $\psi(L) = \psi(L(r)) \cap S$ by 5.3.a and 5.3.b, in which case $L(r, S) = \emptyset$ is decidable since language emptiness is decidable over REG_Σ.

(b) The „canonical" answer would be $r = a^* b^* c^*$, $S = \mathcal{L}\left(\begin{pmatrix}0\\0\\0\end{pmatrix}, \left\{\begin{pmatrix}1\\1\\1\end{pmatrix}\right\}\right)$.

(c) If $\Sigma = \{up, down\}$ then $r = (\Sigma^* up.req\, \Sigma^* up.ack\, \Sigma^*)^*$. Then $\le 20\%$ down time is described by the following semilinear set:

$$S = \mathcal{L}\left(\begin{array}{c} \text{req} \\ \text{ack} \\ \text{up} \\ \text{down} \end{array}\begin{pmatrix}0\\0\\0\\0\end{pmatrix}, \left\{\begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\4\\1\end{pmatrix}\right\}\right).$$