6.1. We show $Sol(t_1 = t_2)$ and $Sol(t_1 < t_2)$ are semilinear by explicitly describing the finite unions of linear sets that specify them.

Each (in)equation $t_1 \square t_2$ can be reduced so that it does not contain the same veriables on both sides of $\square \in \{=, <, >\}$ so one can assume $t_1 \square t_2$ is

$$c + \sum_{0 \le i < k} a_i x_i \;\square\; \sum_{k \le j < n} a_j x_j \quad (\text{for some } k \in \{0, .., n-1\}) \qquad (*)$$

where $c, a_i, a_j \in \mathbb{N} \setminus \{0\}$ and $\{x_0, .., x_{n-1}\} \subseteq Var$.

The assumption is safe since, if one can describe the $\mathbb{N}^n$ solutions of $(*)$ as a semilinear set, the semilinear set in question can be easily extended to one over $\mathbb{N}^{|Var|}$ that includes the variables that cancel out when reducing the Presburger $t_1 \square t_2$.

As for solving $(*)$, let $M = lcm(a_0, .., a_{n-1})$ and define vectors

$$\bar{p}_{ij} := \frac{M}{a_i}\bar{e}_i + \frac{M}{a_j}\bar{e}_j \quad \text{for } 0 \le i < k \text{ and } k \le j < n.$$

Intuitively, vectors $\bar{p}_{ij}$ are periods to either of $t_1 \square t_2$, i.e. $\bar{v} \in \mathbb{N}^n$ is a solution to $(*)$ iff. $\bar{v} + \bar{p}_{ij}$ is also a solution.

Similarly, when $\square \in \{<, >\}$ we must additionally consider the unit vectors as periods since $\bar{v} \in \mathbb{N}^n$ is a solution of

- $t_1 > t_2$ iff. $\bar{v} + \bar{e}_i$ with $0 \le i < k$ is a solution of $t_1 > t_2$
- $t_1 < t_2$ iff. $\bar{v} + \bar{e}_j$ with $k \le j < n$ is a solution of $t_1 < t_2$.

Putting the above together we find $P_= := \{\bar{p}_{ij} \mid 0 \le i < k, k \le j < n\}$, $P_< := P_= \cup \{e_i \mid 0 \le i < k\}$, $P_> := P_= \cup \{e_j \mid k \le j < n\}$ are the <u>finite</u> sets of periods for $Sol(t_1 = t_2)$, $Sol(t_1 < t_2)$, and $Sol(t_1 > t_2)$.

At the same time, the <u>finite</u> sets

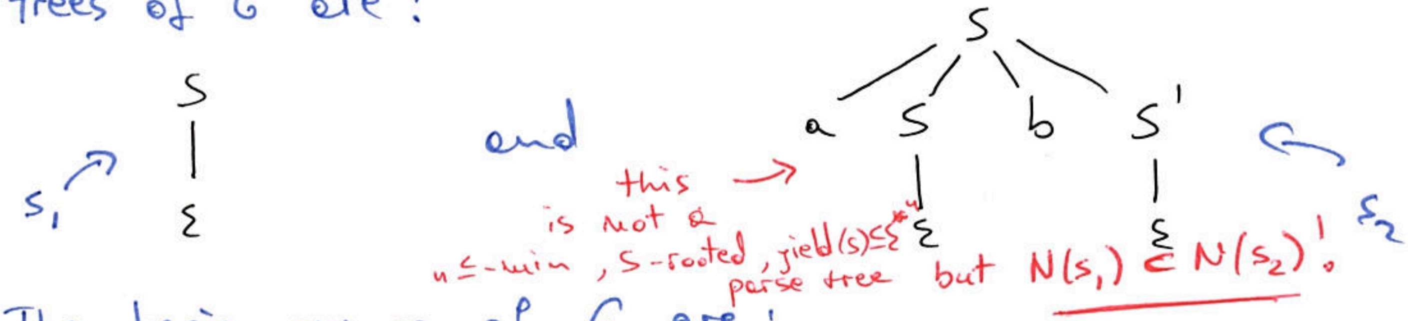
$$B_\square := \{\bar{v} \mid c + \sum_{0 \le i < k} a_i \bar{v}(i) \;\square\; \sum_{k \le j < n} a_j \bar{v}(j), \forall 0 \le i < n.\ \bar{v}(i) < \frac{M}{a_i}\}$$

describe the bases/minimal vectors for $Sol(t_1 \square t_2)$.

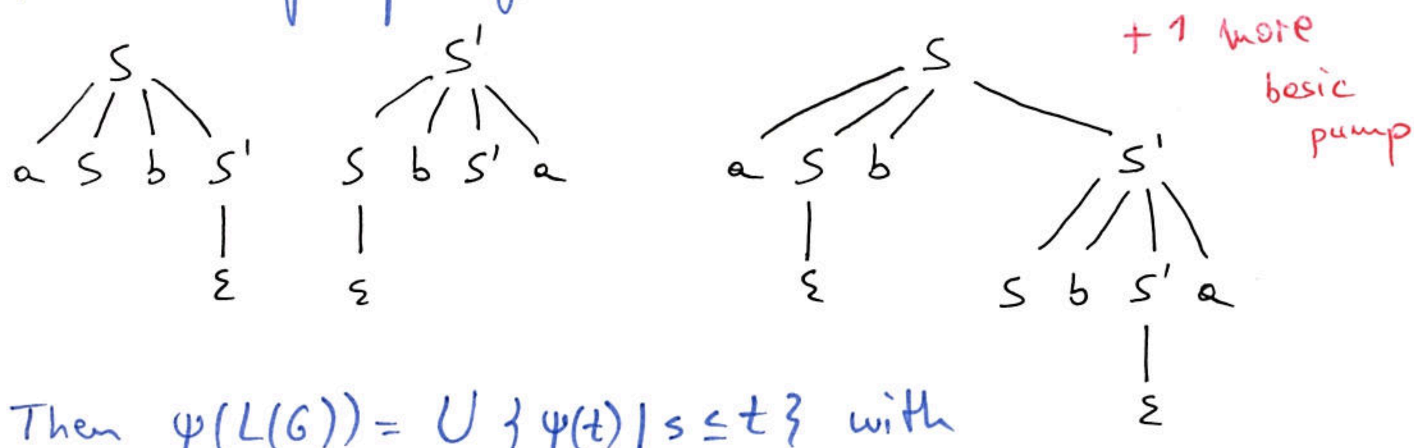
All in all, we find that

$$Sol(t_1 \square t_2) = \bigcup_{\bar{v} \in B_\square} \mathcal{L}(\bar{v}, P_\square).$$

6.2. Let $G = (\{S, S'\}, \{a, b\}, \{S \to aSbS' \mid \varepsilon,\ S' \to SbS'a \mid \varepsilon\}, S)$.
Then the $\leq$-minimal, S-rooted, $yield(s) \subseteq \Sigma^*$ parse trees of G are:



The basic pumps of G are:



Then $\psi(L(G)) = \bigcup_{s \in M} \{\psi(t) \mid s \leq t\}$ with
$M = \{ s \mid \leq\text{-minimal } s,\ root(s) = S,\ yield(s) \subseteq \Sigma^* \}$ i.e.

$$\begin{aligned}
\psi(L(G)) &= \{\psi(t) \mid s_1 \leq t\} \cup \{\psi(t) \mid s_2 \leq t\} \\
&= \mathcal{L}(\Psi(s_1), \{\Psi(u) \mid u \text{ basic pump with } N(u) \subseteq N(s_1)\}) \\
&\cup \mathcal{L}(\Psi(s_2), \{\Psi(u) \mid \text{ -- '' -- '' -- '' } N(u) \subseteq N(s_2)\}) \\
&= \mathcal{L}\left(\binom{0}{0}, \emptyset\right) \cup \mathcal{L}\left(\binom{1}{1}, \left\{\binom{1}{1}, \binom{2}{2}\right\}\right) \\
&= \mathcal{L}\left(\binom{0}{0}, \left\{\binom{1}{1}\right\}\right).
\end{aligned}$$

6.3. (a) Let $A = (\Sigma_A, Q_A, q_{0A}, \to_A, Q_{FA})$ and $B = (\Sigma_B, Q_B, q_{0B}, \to_B, Q_{FA})$ and define

$$C := (\Sigma, Q, q_0, \to, Q_F) \quad \text{where}$$

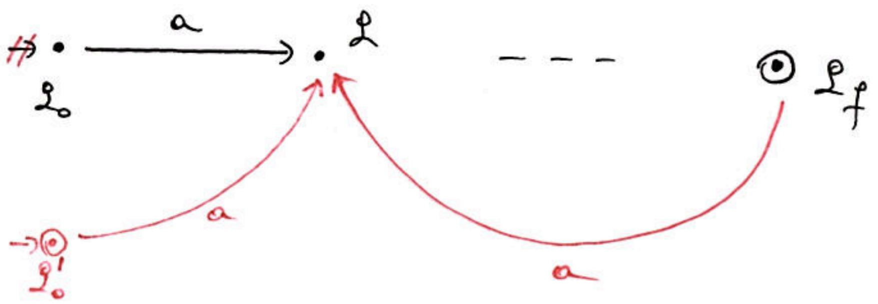
$\Sigma := \Sigma_A \cup \Sigma_B$, $\qquad Q := Q_A \cup Q_B \cup \{q_0\}$,

$Q_F := Q_{FA} \cup Q_{FB}$ and

$\to := \to_A \cup \to_B \cup \bigcup_{a \in \Sigma} \{ q_0 \xrightarrow{a} q \mid q_{0A} \xrightarrow{a} q \text{ or } q_{0B} \xrightarrow{a} q \}$.

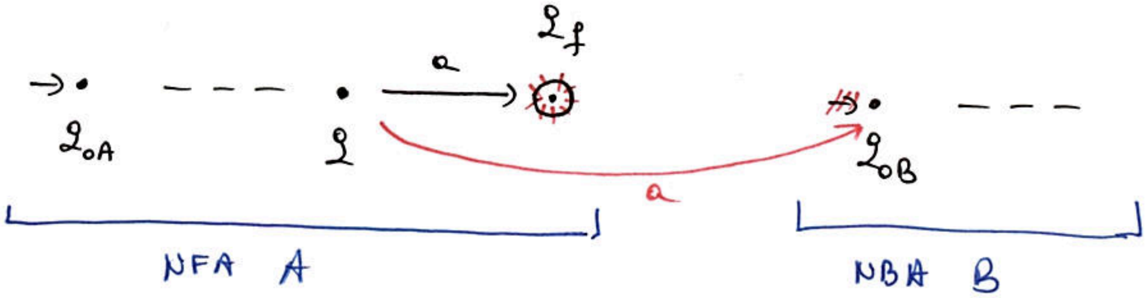
6.3. (b) The construction for the NBA B is the same as for the NFA A^*:

$\forall q \in Q$
$\forall a \in \Sigma$
$\forall q_f \in Q_F$



(c) The construction for NBA C with $L(C) = L(A) . L(B)$ is:

$\forall q \in Q_A$
$\forall a \in \Sigma_A$
$\forall q_f \in Q_{FA}$



6.4. If NBA $A = (\Sigma, Q, q_0, \rightarrow, Q_F)$ then

$$L(A) = \bigcup_{q_f \in Q_F} \underbrace{\{w \in \Sigma^* \mid q_0 \xrightarrow{w} q_f\}}_{V_{q_f}} . \underbrace{\{w \in \Sigma^* \mid q_f \xrightarrow{w} q_f\}}_{W_{q_f}}{}^{\omega}.$$

The fact that the above representation is ω-regular should be clear.

In order to see that the above language contains the Σ^ω projection of every accepting run of A we need only notice that the run will "be" in V_{q_f} until the repeating ~~stat~~ final state q_f is reached, after which the run will always stay in W_{q_f}.