Recapitulation:

- Let θ in LTL in positive normal form
  (constructed from $p, \neg p$ with $p \in P$ and $\wedge, \vee, \bigcirc, \mathcal{U}, \mathcal{R}$)
- A Hintikka set for θ is a subset $M \subseteq FL(\theta)$
  closed under satisfaction of subformulas
  - $\varphi \vee \psi \in M$ implies $\varphi \in M$ or $\psi \in M$
    $\quad\wedge \qquad\qquad\qquad\qquad$ and
  - $\varphi \mathcal{U} \psi \in M$ implies $\psi \in M$ or ($\varphi \in M$ and $\bigcirc(\varphi \mathcal{U} \psi) \in M$).
    $\quad\mathcal{R} \qquad\qquad\qquad$ $\psi \in M$ and ($\varphi \in M$ or $\bigcirc(\varphi \mathcal{R} \psi) \in M$).

M is consistent if there is no $\{p, \neg p\} \subseteq M$.

Set of all consistent Hintikka sets: $\mathcal{H}(\theta)$.

Construct A_θ that accepts precisely models of θ

- States = consistent Hintikka sets
  - ↳ What are the subformulas that hold at this position in the model
  - ↳ Guess them in every step
  - ↳ Consistency:
    - => within Hintikka set:
      automaton does not guess things that are wrong in themselves
    - => with $\bigcirc$:
      if $\bigcirc \varphi$ guessed then φ has to hold at the next state.
- Final states:
  - ↳ construction relies on unrolling of $\mathcal{U}$ and $\mathcal{R}$
    (=> already part of $FL(\theta)$ and Hintikka sets)
  - ↳ until yields accepting states
    - => forbids infinite unrollings (there is a $k \in \mathbb{N}$ for ψ $(\varphi \mathcal{U} \psi)$,

Definition (LTL automaton):

Consider an LTL formula θ in positive normal form.
Let $\varphi_1 \mathcal{U} \psi_1, \dots, \varphi_k \mathcal{U} \psi_k$ all $\mathcal{U}$-formulas in $FL(\theta)$.
Then

$$A_\theta := (\mathcal{H}(\theta), Q_I, \rightarrow, (Q_F^i)_{1 \le i \le k})$$

with

$Q_I := \{ M \in \mathcal{H}(\theta) \mid \theta \in M \}$ // Sets that contain θ.

$Q_F^i := \{ M \in \mathcal{H}(\theta) \mid \varphi_i \mathcal{U} \psi_i \notin M \text{ or } \psi_i \in M \}$

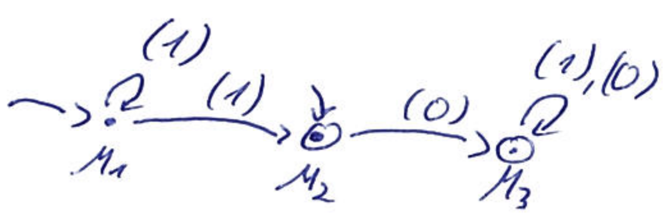
// if the ith until formula needs to be fulfilled then this happens in M.

$M \xrightarrow{a} M'$ if $\{ \psi \in FL(\theta) \mid O\psi \in M \} \subseteq M'$ and $\rho^+(M) \subseteq a$ and $\rho^-(M) \cap a = \emptyset$.

If $FL(\theta)$ does not contain until formulas, pic $Q_F := Q$ as final states.

Example (from yesterday):

$A_{p\mathcal{U}\neg p}$ over $\Sigma = \mathbb{P}(\{p\})$



with

$M_1 = \{ p\mathcal{U}\neg p, p, O(p\mathcal{U}\neg p) \}$

$M_2 = \{ p\mathcal{U}\neg p, \neg p \}$

$M_3 = \emptyset$.

Choose only $\subseteq$-minimal Hintikka sets
↳ More formulas pose more constraints on transitions and successors
↳ Can always be simulated by $\subseteq$-smaller states

<u>Intuitively:</u>

↳ Guess Hintikka set M_0 that contains Θ.

  ⇒ This selects subformulas that also hold at position 0.

↳ If automaton arrives at M, then M contains (potentially negated) propositions p or $\neg p$ and formulas $\bigcirc\psi$

  ⇒ do not have further decompositions

  ⇒ make claims about what has to hold at this position ($\bigcirc\psi$ makes claims about next position)

↳ If automaton takes a transition

  ⇒ only uses alphabet symbols consistent with current propositions (all positive propositions occur, none of the negative propositions is used)

  ⇒ reaches a state consistent with guesses of $\bigcirc$ in previous set (if $\bigcirc\psi \in M$ then $\psi \in M'$).

<u>Theorem:</u>

For every LTL formula Θ, there is an NBA A_Θ with $\mathcal{L}(\Theta) = \mathcal{L}(A_\Theta)$ and $|A_\Theta| \leq 2^{8|\Theta|}$.

<u>Proof:</u>

Wlog., assume Θ is in positive normal form. (conversion may generate linear blow-up).

Let $\varphi_1 \mathcal{U} \psi_1, \ldots, \varphi_k \mathcal{U} \psi_k$ all until-formulas in $FL(\Theta)$.

<u>$\mathcal{L}(\Theta) = \mathcal{L}(A_\Theta)$:</u>

„$\subseteq$" Let $w \in \mathcal{L}(\Theta)$.

Goal is to construct an accepting run of A_Θ on w.

Define for each $i \in \mathbb{N}$ the set

$$M_i := \{\psi \in FL(\Theta) \mid w, i \models \psi\}$$

Then

(a) $M_i \in \mathcal{H}(\Theta)$ // By definition of $\models$ and M_i

(b) $\Theta \in M_0$ // By $w \in \mathcal{L}(\Theta)$ and so $w, 0 \models \Theta$

(c) If $O\psi \in M_i$ then $\psi \in M_{i+1}$ f.a. $i \in \mathbb{N}$ // By definition of $\models$ and M_i

(d) Let $w_i = a$. Then $\beta^+(M_i) \subseteq a$ and $\beta^-(M_i) \cap a = \emptyset$

// By definition of $\models$ and M_i

(e) For all $1 \leq j \leq k$ and all $i \in \mathbb{N}$:

If $\varphi_j \mathcal{U} \psi_j \in M_i$ then there is $i' \geq i$ with $\psi_j \in M_{i'}$.

// If an until-formula holds at some point i, then its right hand side holds at some later moment i'.

// By definition of $\models$ and M_i.

Select the accepting run

$r = M_0 \xrightarrow{a_0} M_1 \xrightarrow{a_1} \ldots$ with $w = a_0 a_1 \ldots$

By (a), $M_0, M_1, \ldots$ is a sequence of states in $\mathcal{A}_\Theta$.

By (b), this sequence starts in $M_0 \in Q_I$.

By (c) and (d), $M_i \xrightarrow{a_i} M_{i+1}$ are valid transitions f.a. $i \in \mathbb{N}$

By (e), run is accepting.

"$\supseteq$" Let $w \in \mathcal{L}(\mathcal{A}_\Theta)$.

We have to show that $w, 0 \models \Theta$.

As $w \in \mathcal{L}(\mathcal{A}_\Theta)$, there is an accepting run

$r = M_0 \xrightarrow{a_0} M_1 \xrightarrow{a_1} \ldots$ of $\mathcal{A}_\Theta$ on w.

By induction on the structure of formulas, we show that

for all $\psi \in FL(\theta)$ and all $i \in \mathbb{N}$ we have

$\psi \in M_i$ implies $w, i \models \psi$.

The above claim follows immediately

(we actually strengthen the induction hypothesis).

IN:

$\psi = p$ If $p \in M_i$ and $M_i \xrightarrow{a_i} M_{i+1}$,

by construction of A_θ we have

$p \in \delta^+(M_i) \subseteq a_i$.

So $w, i \models p$.

$\psi = \neg p$ Similar.

IS: Assume the claim holds for φ and ψ

(on all $i' \in \mathbb{N}$).

$\varphi \wedge \psi$ Let $\varphi \wedge \psi \in M_i$.

By definition of Hintikka sets,

$\varphi \in M_i$ and $\psi \in M_i$.

By the induction hypothesis,

$w, i \models \varphi$ and $w, i \models \psi$.

Thus,

$w, i \models \varphi \wedge \psi$.

$\varphi \vee \psi$ Similar.

$\bigcirc\psi$ Since $\bigcirc\psi \in M_i$, we have $\psi \in M_{i+1}$.

By the induction hypothesis,

$w, i+1 \models \psi$.

Thus,

$w, i \models \bigcirc\psi$.

$\underline{\varphi_j \mathcal{U} \psi_j}$: Let $\varphi_j \mathcal{U} \psi_j \in M_i$ for some $j \in \{1, \dots, k\}$.

By definition of Hintikka sets

(1) $\psi_j \in M_i$ or

(2) $\varphi_j \in M_i$ and $\bigcirc(\varphi_j \mathcal{U} \psi_j) \in M_i$

(we can assume here that $\psi_j \notin M_i$, otherwise we are back to case (1)).

In case (1), we have

$$\omega, i \models \psi_j$$

by the induction hypothesis and thus

$$\omega, i \models \varphi_j \mathcal{U} \psi_j.$$

In case (2), we have

- $\omega, i \models \varphi_j$ by the induction hypothesis and
- $\varphi_j \mathcal{U} \psi_j \in M_{i+1}$ by definition of the transition relation.

We iterate the argument and get

$$\omega, i' \models \varphi_j \quad \text{for } i' = i, i+1, i+2, \dots$$

Furthermore,

$$\varphi_j \mathcal{U} \psi_j \in M_{i'} \quad \text{for } i' = i, i+1, i+2, \dots$$

As the run is accepting, there is some $i' > i$ so that

$$\psi_j \in M_{i'}.$$

An application of the hypothesis yields

$$\omega, i' \models \psi_j.$$

Since furthermore

$$\omega, h \models \varphi_j \quad \text{f.a. } i \leq h < i'$$

we conclude

$$\omega, i \models \varphi_j \mathcal{U} \psi_j.$$

$\varphi_i \, R \, \psi_i$: Similar

Size of the automaton

From ψ in LTL to θ in positive normal form:

$$|\theta| \le 2|\psi|$$

Furthermore, every formula $\varphi \, op \, \psi$ in $FL(\theta)$ yields at most 4 additional formulas (besides subformulas):

$$\varphi \mathcal{U} \psi, \; \bigcirc(\varphi \mathcal{U} \psi), \; \varphi \wedge \bigcirc(\varphi \mathcal{U} \psi), \; \psi \vee (\varphi \wedge \bigcirc(\varphi \mathcal{U} \psi)).$$

Thus:

$$|FL(\theta)| \le 4|\theta| \le 8|\psi|.$$

Automaton A_θ picks all Hintikka subsets of $FL(\theta)$. Their number is bounded by

$$2^{8|\psi|}.$$

$\square$