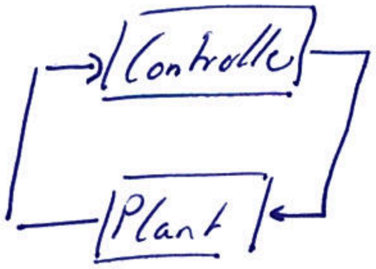
Parity Games

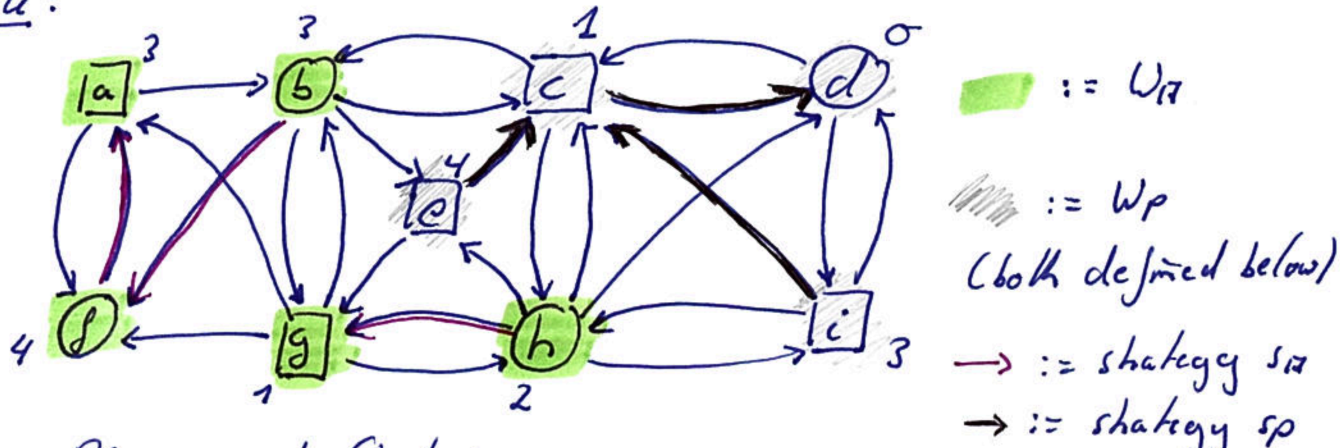
Goal: (1) Solve discrete control problems in reactive systems



(2) Complement parity tree automata (on infinite trees)

Idea:
- 2-Player board game
- Positions labelled by priorities
- Players ∃ and ∀ move a token between positions
- Plays are infinitely long, won by ∃ if highest priority that occurs infinitely often is even.

Example:



Games, Plays, and Strategies

Definition (Parity game, play)

A parity game is a tuple $G = (Pos_\exists, Pos_\forall, \rightarrow, \Omega)$ with

- $Pos_\exists$, $Pos_\forall$ disjoint, not necessarily finite sets of positions.

  Write $Pos := Pos_\exists \uplus Pos_\forall$.
- $\rightarrow \subseteq Pos \times Pos$ is the set of possible moves
- $\Omega : Pos \rightarrow \{0, \ldots, n\}$ assigns priorities to positions.

Additional requirement: no deadlocks

for all $x \in Pos$ there is $y \in Pos : x \rightarrow y$.

↳ A *play* from a given initial position p
is an infinite sequence
$$p = p_0 \to p_1 \to p_2 \to \dots$$
↳ The *play* is *won by player A* if
the largest number that occurs infinitely often in
$$\Omega(p_0), \Omega(p_1), \Omega(p_2), \dots$$
is *even*.

In the example:

$Pos_A = \{b, d, f, h\}$ represented by circles

$Pos_P = \{a, c, e, g, i\}$ represented by boxes.

$\Omega(a) = 3, \Omega(b) = 3, \Omega(c) = 1, \dots$

$\to = \{(a,b), \dots\}$

Interested in whether a player can win the *game*

↳ Formalized via *strategies*:
$$s : Pos^* \times Pos \to Pos$$
Describe the next move of a player,
depending on the moves done so far in the play.

↳ Here we consider *positional strategies*
Next move only depends on current state.

Definition (Strategy):

Let $G = (Pos_A, Pos_P, \to, \Omega)$ be a parity game.

↳ A *positional strategy* for player $i \in \{A, P\}$
is a function
$$s : Pos_i \to Pos.$$
↳ A play $p_0 \to p_1 \to \dots$ is *conform with strategy s for player i*
if for all $p_j \in Pos_i$ we have $p_{j+1} = s(p_j)$.

↳ A strategy s for player i is <u>winning from position p</u> if

player i wins every play that

- starts in p and
- is conform with s.

↳ A <u>winning position for player i</u> is a position p so that there is a winning strategy for player i from p.

The <u>set of winning positions for player i</u> is W_i.

<u>Example:</u>

$W_A = \{a, b, f, g, h\}$

$W_P = \{c, d, e, i\}$

Strategy s_A that is winning from all positions in W_A:

$s_A(f) = a$

$s_A(b) = f$

$s_A(h) = g$

Similarly, s_P is winning from all positions in W_P:

$s_P(e) = c \quad s_P(c) = d \quad s_P(i) = c.$

<u>Lemma:</u>

$W_A \cap W_P = \emptyset.$

<u>Proof:</u>

To the contrary, assume $p \in W_A \cap W_P$.

Then there are winning strategies s_A and s_P for both players A and P from p.

Consider the play that is conform with both strategies.

This means player A and P both win this play.

Then the highest priority that occurs infinitely often is even <u>and</u> odd. A contradiction. $\square$

Assume the winning region involves two nodes.
Then the player has two winning strategies
that may disagree on some nodes.
In general, the number of strategies would grow unbounded
with the number of winning nodes.
Not all of these strategies have to be considered
due to the following.

Lemma:
Let $i \in \{A, P\}$, $G = (Pos_A, Pos_P, \rightarrow, \Omega)$ be a parity game,
and $\mathcal{U} \subseteq Pos$ a set of positions so that from every $p \in \mathcal{U}$
player i has a positional winning strategy s_p.
Then there is a single positional strategy s for player i
that is winning from every $p \in \mathcal{U}$.

Proof:
Collect the positions that are reachable from $p \in \mathcal{U}$
when following s_p:

$$W^p := \{ p' \in Pos \mid \text{there is a sequence } p \rightarrow p_1 \rightarrow \ldots \rightarrow p' \text{ that is conform with } s_p \}.$$

The strategy s we define only for the nodes in

$$W^{\mathcal{U}} := \bigcup_{p \in \mathcal{U}} W^p.$$

For the remaining nodes, s can be chosen arbitrarily.

Trick:
Order the positions $p_0 < p_1 < p_2 < \ldots$ in $\mathcal{U}$.
For each $p \in W^{\mathcal{U}}$, let
$p_i \in \mathcal{U}$ be the element with minimal index i
so that $p \in W^{p_i}$.
Define $s(p) := s_{p_i}(p)$.

One can check that s is winning from all $p_i \in \mathcal{U}$.

$\square$

<u>In the example:</u>

s_P and s_A are of this form.

Note that $W_A \cup W_P = Pos$.

This is not a coincidence – every parity game has this property.

Before we turn to the proof, need a concept.

<u>Definition (Attractor):</u>

Let $G = (Pos_A, Pos_P, \to, \Omega)$ be a parity game and $\mathcal{U} \subseteq Pos$.

The <u>attractor of $\mathcal{U}$ for player A</u> is

$$Attr_A(\mathcal{U}) := \bigcup_{j \in \mathbb{N}} Attr_A^j(\mathcal{U})$$

where

$$Attr_A^0(\mathcal{U}) := \mathcal{U} \quad \text{and} \quad Attr_A^{j+1}(\mathcal{U}) := Attr_A^j(\mathcal{U}) \cup \{p \in Pos_A \mid \exists p' \in Attr_A^j(\mathcal{U}) : p \to p'\} \cup \{p \in Pos_P \mid \forall p' \text{ with } p \to p' : p' \in Attr_A^j(\mathcal{U})\}$$

Similarly: Attractor for player P.

<u>Note:</u>

1) Attractor of set $\mathcal{U}$ for player A

= Positions from which A can force a visit to $\mathcal{U}$, no matter what the opponent does.

2) More precisely: $Attr_A^j(\mathcal{U})$ = can force the visit in $\leq j$ moves.

<u>In the example:</u>

$Attr_A^0(\{f, e\}) = \{f, e\}$

$Attr_A^1(\{f, e\}) = \{f, e, b, h\}$

$Attr_A^2(\{f, e\}) = \{f, e, b, h, a\}$

$Attr_A^3(\{f, e\}) = \{f, e, b, h, a, g\} = Attr_A^4(\{f, e\})$.

An attractor induces a (positional)
<u>attractor strategy</u>:

Map $p \in Pos_A \cap (Attr_A^{j+1}(U) \setminus Attr_A^{j}(U))$

to $p' \in Attr_A^{j}(U)$

so that $p \to p'$.