WS 2016/2017 14.12.2016

Exercises to the lecture
Complexity Theory
Sheet 8

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Dr. Prakash Saivasan Delivery until 10.01.2017 at 10h

Christmas Exercise

Exercise 8.1 (Alternation bounded QBF and collapsing of the polynomial hierarchy)

Consider the following definition:

- $\Sigma_i QBF = \{\psi \mid \psi = \exists\, \overline{x_1}\, \forall\, \overline{x_2} \ldots Q_i\, \overline{x_i}\, \varphi(\overline{x_1}, \ldots, \overline{x_i})$ is true $\}$,
- $\Pi_i QBF = \{\psi \mid \psi = \forall\, \overline{x_1}\, \exists\, \overline{x_2} \ldots Q_i\, \overline{x_i}\, \varphi(\overline{x_1}, \ldots, \overline{x_i})$ is true $\}$,

where $\overline{x_j}$ denotes a finite sequence of variables and Q_i is a quantor. Note that there are at most $i-1$ alternations of quantors.
These *alternation bounded* QBF problems will help us to understand the polynomial hierarchy in more detail:

a) Show that $\Sigma_i QBF$ is in Σ_i^{P} and that $\Pi_i QBF$ is in Π_i^{P}.

b) Prove that $\Sigma_i QBF$ is Σ_i^{P}-hard with respect to polytime reductions and that $\Pi_i QBF$ is Π_i^{P}-hard with respect to polytime reductions.
Hint: Take an arbitrary language in Σ_i^{P} and reduce it to $\Sigma_i QBF$. Note that we showed that QBF is PSPACE-complete. Extract the idea from this proof.

Exercise 8.2 (co-Oracles)
Let $\mathcal{C}$ be a complexity class. Show that using oracles for $\mathcal{C}$ is equivalent to using oracles for co-$\mathcal{C}$:

a) Prove that $\mathsf{NP}^B = \mathsf{NP}^{\bar{B}}$ for any problem B in $\mathcal{C}$.

b) Conclude that we have: $\mathsf{NP}^{\mathcal{C}} = \mathsf{NP}^{\text{co-}\mathcal{C}}$.

Exercise 8.3 (Minimal Boolean formulas)
Two Boolean formulas are called **equivalent** if they have the same value on any assignment to the variables. A formula φ is called **minimal** if there is no smaller formula that is equivalent to φ.
Consider the problem:

$$MIN = \{\varphi \mid \varphi \text{ is minimal}\}.$$

a) Show that deciding whether two formulas are equivalent is in co-NP.

b) Prove that the co-problem $NOTMIN = \{\varphi \mid \varphi \text{ is not minimal}\}$ is in $\mathsf{NP}^{\mathsf{NP}}$.

c) Conclude that MIN is a problem in Π_2^{P}.

Wish you all a merry Christmas and a very happy new year. Enjoy your vacation!

Delivery until 10.01.2017 at 10h into the box next to room 343 in the Institute for Theoretical Computer Science, Muehlenpfordstrasse 22-23