Exercises to the lecture
Complexity Theory
Sheet 13

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Exercise 13.1 (Maximal satisfiability)

Consider the following problem, called *MAXSAT*:
Input: A formula $\varphi = \bigwedge_{i=1}^{m} C_i$, where the C_i are the clauses, and $k \in \mathbb{N}$.
Parameter: k.
Question: Does there exist a variable assignment that satisfies at least k clauses ?

In this exercise we will construct a kernelization for *MAXSAT* and show that the kernel has size bounded by $\mathcal{O}(k^2)$. To this end, recall the following:

- A **literal** is either a variable or a negated variable.
- A **clause** is a disjunction of literals.
- The **size** of a CNF-formula φ is the sum of the numbers of literals used in the clauses of φ.

Let (φ, k) be a problem instance. Our first reduction step is to delete all *trivial* clauses. A clause in φ is called **trivial** if it contains a variable and its negation.

a) Show that, by removing all trivial clauses, we can reduce (φ, k) to a problem instance (φ_n, k') so that: $k' \leq k$ and $(\varphi_n, k') \in MAXSAT$ if and only if $(\varphi, k) \in MAXSAT$.

For a further reduction, we look at **long** clauses: these are clauses that contain more than k' literals.

b) Prove the following: If φ_n contains more than k' long clauses, then $(\varphi_n, k') \in MAXSAT$.
Hint: We only need one true literal for a clause to be satisfied.

c) Denote by $(\varphi_s, \hat{k})$, where $\hat{k} = k' - t$, the instance that we get if we remove all t long clauses from (φ_n, k'). Show that we have: $(\varphi_n, k') \in MAXSAT$ if and only if $(\varphi_s, \hat{k}) \in MAXSAT$.

Hence, we obtain an instance that only consists of clauses of size at most k'. Now we show that the size of such an instance is also bounded by the parameter:

d) If $(\varphi_s, \hat{k})$ has more than $2\hat{k}$ clauses, then show that $(\varphi_s, \hat{k})$ is in *MAXSAT*.
Hint: Look at any variable assignment and its complement.

Finally, we can construct a kernelization for *MAXSAT*:

e) Summarize the reduction steps in an algorithm and show that the size of the kernel (the size of the formulas, obtained from the reduction steps) is bounded by $\mathcal{O}(k^2)$.

Exercise 13.2 (Karp's reduction)

Consider the problem $3SAT$:
Given: A formula ψ in CNF so that the clauses of ψ contain at most 3 literals.
Question: Is ψ satisfiable ?

The reduction of Karp maps SAT-instances φ to $3SAT$-instances by the following rule: if $C = \ell_1 \vee \cdots \vee \ell_t$ is a clause of φ, it is mapped to the formula:

$$(\ell_1 \vee \ell_2 \vee z_1) \wedge (\neg z_1 \vee \ell_3 \vee z_2) \wedge (\neg z_2 \vee \ell_4 \vee z_3) \wedge \cdots \wedge (\neg z_{t-3} \vee \ell_{t-1} \vee \ell_t)$$

This creates a $3SAT$-instance.

a) If φ denotes a SAT-instance and f denotes Karp's reduction, show that we have:

$$\varphi \in SAT \text{ if and only if } f(\varphi) \in 3SAT.$$

Hint: You may use the concept of resolution to show this.

Now consider the parametrized problem *WEIGHTEDSAT*:
Given: A formula φ in CNF and a natural number k.
Parameter: $k \in \mathbb{N}$.
Question: Is there a satisfying assignment for φ that evaluates *exactly* k variables to 1.

The problem *WEIGHTED*$3SAT$ is defined similarly.

b) Argue why Karp's reduction is not a parametrized many-one reduction from *WEIGHTEDSAT* to *WEIGHTED3SAT*.

Exercise 13.3 (Reduction to FPT)

Let $L, L' \subseteq \Sigma^* \times \mathbb{N}$ be parametrized languages so that L reduces to L' by a parametrized many-one reduction and L' is FPT. Show that also L is FPT.

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