Exercises to the lecture
Complexity Theory
Sheet 9

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Thomas Haas Delivery until 21.02.2022 at 15:00

Exercise 9.1 (Matrix Decomposition)

Let $M \in \mathbb{N}^{m \times \ell}$ be a matrix with m rows and ℓ columns over the natural numbers. We define the *grand sum* of M, denoted by $\mathrm{gs}(M)$, to be the sum over all entries of M. Let $M_1 \in \mathbb{N}^{r_1 \times \ell}$ and $M_2 \in \mathbb{N}^{r_2 \times \ell}$ be *submatrices* of M. This means that M_1 and M_2 are matrices that are build by putting together r_1 (r_2 respectively) rows of M. The matrices M_1 and M_2 are said to *decompose* M if $r_1 + r_2 = m$. In other words, putting the rows of M_1 and M_2 together in the correct order rebuilds M.
In the following problem we compute the minimal number of submatrices of M that are needed to decompose M in such a way that each of the submatrices M_i satisfies $\mathrm{gs}(M_i) \leq D$ for a given bound $D \in \mathbb{N}$.

Matrix Decomposition	
Input:	A matrix $M \in \mathbb{N}^{m \times \ell}$ and a bound $D \in \mathbb{N}$.
Parameter:	The number of rows m.
Question:	Find the minimal $t \in \mathbb{N}$ such that M can be decomposed into submatrices $M_1, \ldots, M_t$ with $M_i \in \mathbb{N}^{r_i \times \ell}$ and $\mathrm{gs}(M_i) \leq D$.

Given an algorithm for the problem running in time $2^m \cdot n^{\mathcal{O}(1)}$.

Hint: Use the fast subset convolution.

Exercise 9.2 (Packing Product)

The *packing product* of two functions $f, g : \mathcal{P}(V) \to \mathbb{Z}$ is a function $(f *_p g) : \mathcal{P}(V) \to \mathbb{Z}$ such that

$$(f *_p g)(X) = \sum_{\substack{A,B \subseteq X \\ A \cap B = \emptyset}} f(A) \cdot g(B).$$

Show that all the 2^n values of the packing product can be computed in time $2^n \cdot n^{\mathcal{O}(1)}$, where $n = |V|$.

Hint: Represent the packing product in terms of the subset convolution.

Exercise 9.3 (Separators)

Let $G = (V, E)$ be a graph and $A, B \subseteq V$. Prove that (A, B) is a separation of G if and only if $A \cup B = V$ and $\delta(A) \subseteq A \cap B$.

Exercise 9.4 (Treewidth)

A *forest* is an undirected graph the connected components of which are all trees. Phrased differently, a forest is a disjoint union of trees.

Determine the treewidth of a forest.

Exercise 9.5 (Treewidth of Cliques)

Let G be a graph and $(T, \{X_t\}_{t\in V(T)})$ a tree decomposition of G. Show that each clique in G is contained in a single bag of $(T, \{X_t\}_{t\in V(T)})$.

Derive that $tw(G) \geq \omega(G) - 1$, where $\omega(G)$ is the maximal size of a clique in G.

Hint: Let C be the set of vertices of a clique and let st be an edge in the tree T. Use the separation lemma to show that either $C \subseteq V_s$ or $C \subseteq V_t$, where $V_s = \bigcup_{u\in T_s} X_u$ and $V_t = \bigcup_{u\in T_t} X_u$. Like in the separation lemma, the trees T_s and T_t are obtained from removing the edge st from T.

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