2. Coverability is in EXPSPACE, Rackoff '78

Coverability:

Given: Petri net $N = (S, T, W)$ and markings $M_1, M_2 \in \mathbb{N}^S$.

Problem: $\exists M \in R(M_1): M \geq M_2$?

Next week: Lower bound

Coverability is EXPSPACE-*hard*, Lipton '76.

Now: Upper bound

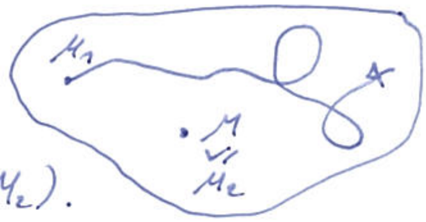
Coverability is *in* EXPSPACE, Rackoff '78.

This means there is an algorithm that
- decides coverability and
- needs only exponential space (in the size of the input)

Together: Coverability is EXPSPACE-complete.

Approach due to Rackoff:

- Guess a path from M_1 to M
- Terminate if the path becomes too long
- Too long = longer than 2^{2^n}, $n = \text{size}(N, M_1, M_2)$.



Why does the procedure only need exponential space?

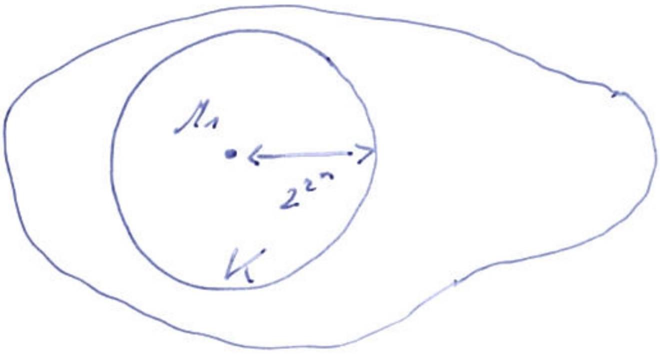
↳ Length of the path (up to 2^{2^n}) can be stored in binary with

$$\log 2^{2^n} = 2^n \text{ bits.}$$

↳ To continue the path, we only need the *last marking*

↳ Given that the length of the path is 2^{2^n},
no place can have more than 2^{2^n} tokens.
Again use binary representation.

<u>Statement of Rackoff's theorem:</u>



- The markings coverable from M_1 form the ball $K\downarrow$.
- Phrased differently, every marking coverable from M_1 is covered in K.

<u>Note:</u>

- The algorithm is non-deterministic.
- This means we have shown that coverability is in NEXPSPACE.
- But NEXPSPACE = EXPSPACE by Savitch's theorem.
- This yields a deterministic algorithm in EXPSPACE.

<u>Challenge:</u>
Establish property of short paths:
If $M_2 \in R(M_1)\downarrow$,
then there is $\tau \in T^*$ with $|\tau| \leq 2^{2^n}$
so that $M_1[\tau\rangle M$ with $M \geq M_2$.

<u>Definition (i-r-bounded, i-covering):</u>
h = dimension of PN N.

- Let $W \in \mathbb{Z}^h$ and $i \in [0, h]$,

↳ Call W <u>i-bounded</u>,
if for all $1 \leq j \leq i$ we have
$0 \leq W(j)$

$$W = \begin{pmatrix} i \\ \\ \end{pmatrix}\begin{array}{l} \} \mathbb{N} \\ \} \mathbb{Z} \end{array}$$

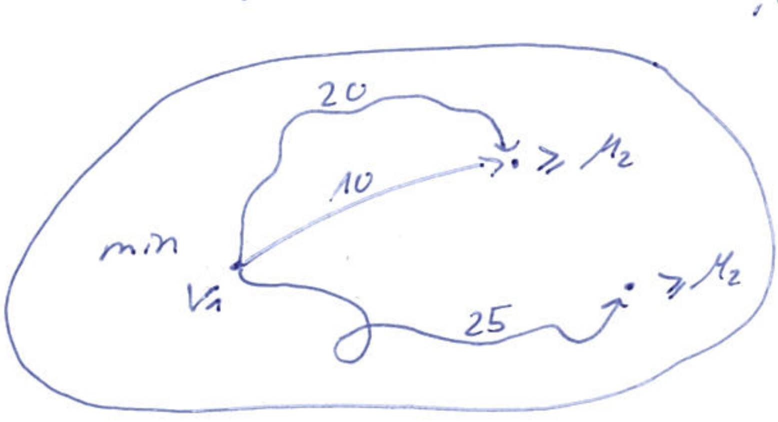
↳ Let $r \in \mathbb{N} \setminus \{0\}$.
Call W <u>i-r-bounded</u>,
if for all $1 \le j \le i$ we have
$$0 \le W(j) < r.$$

- Consider a sequence of (generalized) markings
$$\sigma = W_1 W_2 \dots W_m \in (\mathbb{Z}^k)^*$$
↳ The sequence is called <u>i-bounded</u> / <u>i-r-bounded</u>, if this holds for every marking.

↳ The sequence is called <u>i-covering</u> if
$$W_m(j) \ge M_2(j) \quad \text{for all } 1 \le j \le i.$$

<u>Definition</u> (Worst-case bounds on shortest covering sequences):
Let $V \in \mathbb{Z}^k$. Define
$$m(i, V) := \begin{cases} \min\{ |\sigma| \mid \sigma \in (\mathbb{Z}^k)^* \text{ is } i\text{-bounded and } i\text{-covering in } (N, V)\}, & \text{if exists} \\ 0, & \text{otherwise.} \end{cases}$$
initial marking

<u>Illustration</u>

$m(i, V_1) =$  $= 10.$

Moreover, define
$$f(i) := \max\{ m(i, V) \mid V \in \mathbb{Z}^k \}.$$

<u>Intuition</u>:
- $f(i)$ = maximal length of a shortest covering sequence.

To be more precise:

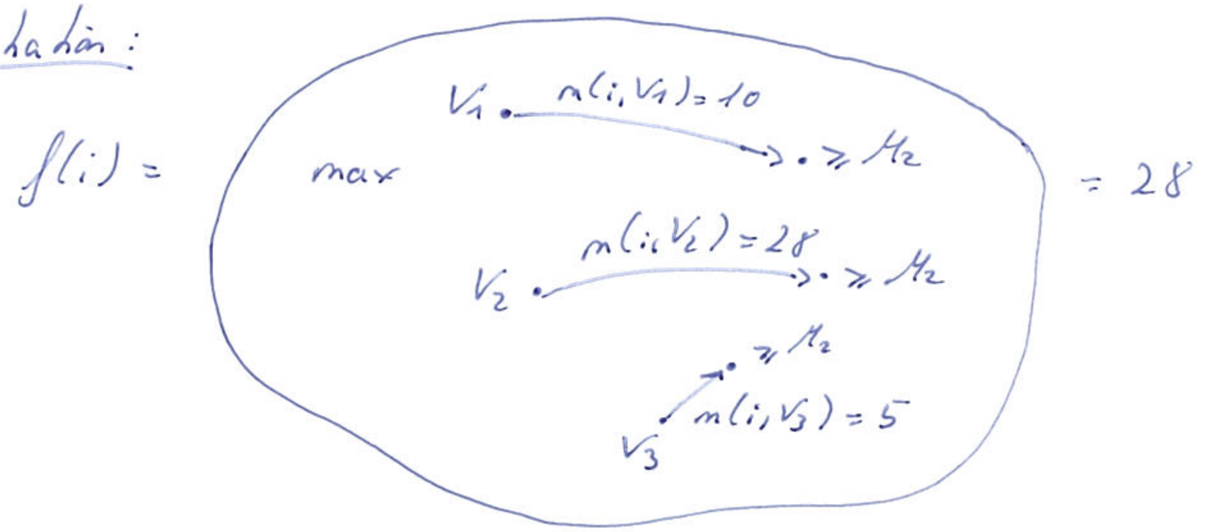
↳ Consider all initial markings

↳ How long can it take

↳ to cover $M_2(j)$ for all $1 \leq j \leq i$ and

↳ maintain the first i-dimensions positive.

<u>Illustration:</u>



<u>Lemma:</u>

Let $V \in \mathbb{Z}^k$ and $M_2 \in \mathbb{N}^k$. Let N be of dimension k.

Then $M_2 \in R(N, V)\downarrow$ iff
there is a k-bounded and k-covering path.

<u>Proof:</u>

"⇒" Since $M_2 \in R(N, V)\downarrow$, there is a sequence
$$V, V_1, \dots, V_m \geq M_2.$$

↳ Since this is a PN-path,
all vectors are positive in all entries.
Hence, the sequence is k-bounded.

↳ Moreover, $V_m \geq M_2$ by assumption.
Hence, the sequence is k-covering.

"⇐" Consider a k-bounded and k-covering path.

↳ Since k-bounded, it is a PN-path.

↳ Since h-covering, we have $V_{last} \geq M_2$.

Hence, $M_2 \in R(N, V)\downarrow$ $\square$

Goal: · Determine an upper bound on $f(h)$

· Then we have

$$f(h) \geq m(h, M_1) \text{ by definition.}$$

· But $m(h, M_1)$ is precisely the length of the shortest path that M_1 needs to cover M_2.

Approach: Use induction.