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Basics of Well-Quasi-Orderings

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The algorithm deciding termination on Petri nets essentially relies on the following two facts:

1. Transitions in Petri nets are monotonic: if $M_1 \xrightarrow{\sigma} M_2$ and $M_1' \geq M_1$ then $M_1' \xrightarrow{\sigma} M_2' \geq M_2$ (larger markings can simulate smaller ones).
2. The state space $\mathbb{N}^k$ of a Petri net is a *well-quasi-ordering (wqo)*.

It turns out that the algorithm can be adapted to decide termination on an entire class of systems satisfying these two properties (we will call them *well structured transition systems* and study them in great detail).

The goal of this note is to lay down the basic theory of wqo, which we will extensively exploit later in the lecture.

1 Characterisations of wqo

Definition 1. We say $(Q, \leq)$ is a *quasi-ordering (qo)* if $\leq \subseteq Q \times Q$ is reflexive and transitive. We call $P \subseteq Q$ an *antichain* if $\forall x, y \in P : x \neq y \Rightarrow x \not\leq y$.

In the following, let $(Q, \leq)$ be a qo.

Definition 2. A pair (a, b) of elements of Q is called *increasing* if $a \leq b$ and *strictly increasing* if additionally $b \not\leq a$; notation $a < b$. *Decreasing* and *strictly decreasing* are defined analogously. An element $a \in P \subseteq Q$ is said to be *minimal* (resp. *maximal*) in P if there is no $b \in P$ with $b < a$ (resp. $b > a$).

We denote the *upward closure* of $P \subseteq Q$ as $P\!\uparrow := \{x \in Q \mid \exists y \in P : y \leq x\}$. The *downward closure* of P is $P\!\downarrow := \{x \in Q \mid \exists y \in P : x \leq y\}$. We say P is *upward closed* if $P = P\!\uparrow$ and *downward closed* if $P = P\!\downarrow$.

Definition 3. A sequence $(a_i)_{i \in \mathbb{N}}$ is called *ascending* or *strictly ascending* if $a_i \leq a_{i+1}$ respectively $a_i < a_{i+1}$ for every $i \in \mathbb{N}$. *Descending* and *strictly descending* are defined analogously.

Definition 4. A sequence $(a_i)_{i \in \mathbb{N}}$ over Q is called *good* if there are $i < j$ with $a_i \leq a_j$ otherwise it is called *bad*. A *well-quasi-ordering (wqo)* is a qo over which every infinite sequence is good.

There are other equivalent ways of defining wqo, and each makes explicit an important aspect of the wqo condition.

Theorem 1 (Characterisation of wqo)**.** *The following statements are equivalent:*

1. *Q is a wqo.*
2. *Every sequence $(a_i)_{i\in\mathbb{N}}$ over Q has an infinite ascending subsequence.*
3. *Every strictly descending sequence over Q is finite (i.e. Q is* well-founded*) and every antichain of Q is finite.*
4. *For every $P \subseteq Q$ there is a finite set $P' \subseteq P$ of elements minimal in P such that $P \subseteq P'{\uparrow}$.*

Proof. "1 ⇒ 2": Let $A = (a_i)_{i\in\mathbb{N}}$ be an infinite sequence over Q. Consider the subsequence $A' = a_{\phi_0}a_{\phi_1}\cdots$ of elements in the sequence that are not dominated by a successor, formally $\nexists i > \phi_k : a_i \geq a_{\phi_k}$. Since Q is a wqo, A' has to be finite, i.e. $A' = a_{\phi_0}\cdots a_{\phi_k}$. Thus, we will find an infinite ascending subsequence starting with a_{ϕ_k+1}.

"2 ⇒ 3": Assume there is an infinite strictly descending sequence $(a_i)_{i\in\mathbb{N}}$. By induction, $a_i \not\leq a_j$ for all $i < j$. Contradiction to $(a_i)_{i\in\mathbb{N}}$ is good.

Assume there is an infinite antichain A. There is an infinite sequence $(a_i)_{i\in\mathbb{N}}$ over A that is good by assumption. Contradiction to A antichain.

"3 ⇒ 4": Let $P \subseteq Q$ and set $P_0 = P$. If $P_i \neq \emptyset$ choose $a_{i,0} \in P_i$. If $a_{i,j}$ is not minimal in P_i, let $a_{i,j} > a_{i,j+1} \in P_i$. As every strictly descending sequence is finite, there is a_{i,k_i} that is minimal in P_i. Let $P_{i+1} = P_i \setminus a_{i,k_i}{\uparrow}$. As $\{a_{0,k_0}, a_{1,k_1}, \ldots\}$ forms an antichain, it is finite. Thus, $P_n = \emptyset$ for some $n \in \mathbb{N}$ and $\{a_{0,k_0}, \ldots, a_{n-1,k_{n-1}}\}$ fulfils our needs.

"4 ⇒ 1": Let $(a_i)_{i\in\mathbb{N}}$ be an infinite sequence over Q. By assumption, there is a finite set $P' = \{a_{\phi_0}, \ldots, a_{\phi_k}\}$ such that $\{a_i \mid i \in \mathbb{N}\} \subseteq P'{\uparrow}$ and $\phi_i < \phi_{i+1}$ for all i wlog. We have $a_{\phi_i} \leq a_{\phi_k+1}$ for some i. ☐

Remark 1. *"3 ⇒ 1" and even the stronger "3 ⇒ 2" can alternatively be proven as an application of Ramseys theorem.*

For example, $(\mathbb{N}, \leq)$ is a wqo: it has no antichains and is well-founded. However, $(\mathbb{Z}, \leq)$ is a qo but no wqo because it is not well-founded. The *discrete ordering* $(X, =)$ is a qo for every set; it is a wqo iff X is finite.

Lemma 1. *Let $(Q, \leq)$ be a wqo. Then if $Q' \subseteq Q$ then $(Q', \leq)$ is a wqo. Moreover, if for a quasi-ordering $\sqsubseteq$ on Q, for all $q, q' \in Q$ we have $q \leq q' \Rightarrow q \sqsubseteq q'$, then $(Q, \sqsubseteq)$ is a wqo.*

Proof. A bad sequence of $(Q', \leq)$ or of $(Q, \sqsubseteq)$ is a bad sequence of $(Q, \leq)$. ☐

2 Constructing wqo

By constructing wqo we mean that we can show that from some wqo we can build other wqo using some common constructions. The first we consider is the Cartesian product.

Lemma 2. *Let* $(Q_1, \leq_1)$, $(Q_2, \leq_2)$ *be wqo. Then* $(Q_1 \times Q_2, \leq)$ *with* $(p_1, p_2) \leq (q_1, q_2) \Leftrightarrow p_1 \leq_1 q_1 \wedge p_2 \leq_2 q_2$ *is a wqo.*

Proof. $Q_1 \times Q_2$ is a qo. Let $((a_i, b_i))_{i\in\mathbb{N}}$ be a sequence over $Q_1 \times Q_2$. By item 2 in Theorem 1, there is an infinite ascending subsequence $a_{\phi_0} \leq_1 a_{\phi_1} \leq \cdots$ of $(a_i)_{i\in\mathbb{N}}$. Because $(b_{\phi_i})_{i\in\mathbb{N}}$ is good by assumption, there are $i < j$ such that $b_{\phi_i} \leq_2 b_{\phi_j}$, and thus, $(a_{\phi_i}, b_{\phi_i}) \leq (a_{\phi_j}, b_{\phi_j})$. □

Corollary 1 (Dickson's Lemma)**.** *For every* $k \in \mathbb{N}$, $(\mathbb{N}^k, \leq_k)$ *is a wqo, where* $(n_1, \ldots, n_k) \leq_k (m_1, \ldots, m_k)$ *if for each* $i = 1, \ldots, k$ *we have* $n_i \leq m_i$.

We now study the wqo properties of domains built from other wqo. Say you have a wqo $(Q, \leq)$, we will prove that a certain operation F will give rise to another wqo $(F(Q), F(\leq))$: for example a generalisation of Dickson's lemma (that follows from Lemma 2) is an instance of this scheme by setting $F(Q) = Q^k$ and $F(\leq) = \{(\vec{x}, \vec{y}) \mid x_i \leq y_i, i = 1, \ldots, k\}$. We will see the case of finite subsets of Q ($F(Q) = \mathcal{P}_f(Q)$), finite trees labelled by elements of Q ($F(Q) = \mathcal{T}(Q)$) and finite words over Q ($F(Q) = Q^*$). In each case the induced $F(\leq)$ is obtained by introducing a concept of *embedding*: a structure s_1 in $F(Q)$ is embedded in another s_2 if there is an injective function relating the components of s_1 to components of s_2 so that the underlying ordering $\leq$ is preserved by the mapping. Let us formalise this for each case.

Definition 5. We define $\mathcal{P}_f(Q) := \{P \subseteq Q \mid P \text{ is finite}\}$. Let $P, P' \in \mathcal{P}_f(Q)$, a *subset embedding* from P to P', is an injective function $\varphi\colon P \to P'$ such that for all $x \in P$, $x \leq \varphi(x)$. The *subset embedding ordering* is the quasi-ordering $\sqsubseteq_{\mathcal{P}}$ over $\mathcal{P}_f(Q)$ where $P \sqsubseteq_{\mathcal{P}} P'$ iff there is a subset embedding from P to P'.

Take for example the wqo $(\mathbb{N}, \leq)$. We have that $\{1, 3, 5\} \sqsubseteq_{\mathcal{P}} \{0, 3, 4, 5, 20\}$ but $\{4, 7\} \not\sqsubseteq_{\mathcal{P}} \{1, 2, 5, 6\}$ and $\{1, 2, 3, 4\} \not\sqsubseteq_{\mathcal{P}} \{10, 20, 30\}$.

Note that for all $A \in \mathcal{P}_f(Q)$, we have $\emptyset \sqsubseteq_{\mathcal{P}} A$ since the empty function $\perp\colon \emptyset \to A$ is a subset embedding. When the underlying ordering is $(Q, =)$ then subset embedding is simply set inclusion.

Lemma 3. *If* $(Q, \leq)$ *is a wqo, then* $(\mathcal{P}_f(Q), \sqsubseteq_{\mathcal{P}})$ *is a wqo.*

Proof. Clearly, it is a qo. Assume $\mathcal{P}_f(Q)$ has bad sequences. We will construct a "lexicographically minimal" bad sequence: Choose $A_0 \in \mathcal{P}_f(Q)$ such that it is the first term in a bad sequence and $|A_0|$ is minimal. If we chose $A_0, \ldots, A_k$, choose A_{k+1} such that $A_0 \cdots A_{k+1}$ is the beginning of a bad sequence and $|A_{k+1}|$ is minimal. The so constructed $(A_i)_{i\in\mathbb{N}}$ is a bad sequence.

No A_i can be empty, otherwise $A_i = \emptyset \sqsubseteq_{\mathcal{P}} A_{i+1}$, so we can pick an element from each set in the sequence: for each $i \in \mathbb{N}$, pick $a_i \in A_i$ and let $B_i = A_i \setminus a_i$. We show that $(B, \leq|_{B\times B})$ is a wqo where $B = \{B_i \mid i \in \mathbb{N}\}$: Let $(B_{f(i)})_{i\in\mathbb{N}}$ be a sequence over B. Let $k \in \mathbb{N}$ such that $f(k) = \min f(\mathbb{N})$, in particular, $f(i) \geq f(k)$ for $i \geq k$. Consider the sequence

$$A_0 \ \cdots \ A_{f(k)-1} \ B_{f(k)} \ B_{f(k+1)} \ \cdots$$

As $|B_{f(k)}| < |A_{f(k)}|$, this sequence cannot be bad as this would contradict the choice of $A_{f(k)}$. Furthermore, as $A_i \leq B_{f(j)}$ implies $A_i \leq A_{f(j)}$ for $i < f(k)$, $j \geq k$ and $(A_i)_{i\in\mathbb{N}}$ is bad, there have to be $k \leq i < j$ such that $B_{f(i)} \leq B_{f(j)}$. Thus, $\big(B_{f(i)}\big)_{i\in\mathbb{N}}$ is good and B wqo.

By Lemma 2, $Q \times B$ is wqo. Therefore $((a_i, B_i))_{i\in\mathbb{N}}$ is good, i.e. there are $i < j$ with $(a_i, B_i) \leq (a_j, B_j)$, which means there is a subset embedding $\varphi\colon B_i \to B_j$ which we can extend to map $\varphi(a_i) = a_j$ proving $A_i \sqsubseteq_{\mathcal{P}} A_j$. Contradiction with the badness of $(A_i)_{i\in\mathbb{N}}$. □

A *graph* G consists of a finite set $V(G)$ of *vertices* and a set $E(G) \subseteq V(G) \times V(G)$ of *edges*. A *tree* T is a graph that has a *root* $\rho(T) \in V(T)$ and where for every $v \in V$ there is a unique path (defined as usual) from $\rho(T)$ to v in T. In a tree T, we say $v \in V(T)$ is the child of $v' \in V(T)$ if $(v', v) \in E(T)$. We say v' is an *ancestor* of v in T if v' is in the path from the root to v.

Definition 6. Let $(X, \leq)$ be a qo. A *X-labelled tree* T is a tree equipped with a labelling function $\lambda(T) : V(T) \to X$, associating each node to a label in X. We denote the set of labelled trees over X as $\mathcal{T}(X)$.

For trees $T, T' \in \mathcal{T}(X)$, a *tree embedding* from T to T' is an injective function $\varphi\colon V(T) \to V(T')$ such that for all $v \in V(T)$:

1. $\lambda(T)(v) \leq \lambda(T')(\varphi(v))$, and
2. $v' \in V(T)$ is an ancestor of v in T, if and only if $\varphi(v')$ is an ancestor of $\varphi(v)$ in T'.

We define the ordering $\sqsubseteq_{\mathcal{T}}$ on $\mathcal{T}(X)$ so that, for trees $T, T' \in \mathcal{T}(X)$ we have $T \leq T'$ if there exists a tree embedding from T to T'.

By adapting the same proof technique we used for proving Lemma 3 (and using the lemma as well) we can prove that trees with wqo labels form a wqo. The result is named after Kruskal, who proved it in a paper published in 1960. Here we present a simpler proof due to Nash-Williams.

Theorem 2 (Kruskal's theorem)**.** *If $(X, \leq)$ is a wqo so is $(\mathcal{T}(X), \leq)$.*

Proof. Clearly, $(\mathcal{T}(X), \leq)$ is a qo. Assume there are bad sequences. Like in Lemma 3, we construct a bad sequence $(T_i)_{i\in\mathbb{N}}$ such that $|V(T_i)|$ is minimal in every step. Since it is a bad sequence, no tree in it is empty. Let B_i be the (finite) set of subtrees of T_i rooted at the children of $\rho(T_i)$ and let $B = \bigcup_{i\in\mathbb{N}} B_i$.

We show that $(B, \leq|_{B\times B})$ is a wqo: Consider a sequence $(R_i)_{i\in\mathbb{N}}$ over B. By construction, there is $f\colon \mathbb{N} \to \mathbb{N}$ such that $R_i \in B_{f(i)}$ for all $i \in \mathbb{N}$. Similar to Lemma 3, let $k \in \mathbb{N}$ such that $f(k) = \min f(\mathbb{N})$, in particular, $f(i) \geq f(k)$ for $i \geq k$. Consider the sequence

$$T_0 \ \cdots \ T_{f(k)-1} \ R_k \ R_{k+1} \ \cdots$$

As $|R_k| < |T_{f(k)}|$, this sequence cannot be bad as this would contradict the choice of $T_{f(k)}$. Furthermore, as $T_i \leq R_j$ implies $T_i \leq T_{f(j)}$ for $i < f(k)$, $j \geq k$ and $(T_i)_{i\in\mathbb{N}}$ is bad, there have to be $k \leq i < j$ such that $R_i \leq R_j$. Thus, $(R_i)_{i\in\mathbb{N}}$ is good and B is a wqo.

By Lemmas 2 and 3, $X \times \mathcal{P}_f(B)$ is a wqo. Thus, $((\lambda(T_i)(\rho(T_i)), B_i))_{i\in\mathbb{N}}$ is good and there are $i < j$ with $\lambda(T_i)(\rho(T_i)) \leq \lambda(T_j)(\rho(T_j))$ and $B_i \sqsubseteq_{\mathcal{P}} B_j$.

By the latter, there is a subset embedding $\varphi : B_i \to B_j$, i.e. $R \sqsubseteq_{\mathcal{T}} \varphi(R)$ for all $R \in B_i$, which in turn means that there is a tree embedding ψ_R of R into $\varphi(R)$. Define an embedding ψ of T_i into T_j as follows:

$$\psi(v) = \begin{cases} \rho(T_j) & \text{if } v = \rho(T_i) \\ \psi_R(v) & \text{if } v \in R, \text{ for some } R \in B_i \end{cases}$$

Note that every $v \in V(T_i)$ is in one an only one $R \in B_i$ or is the root of T_i. Now from $\lambda(T_i)(\rho(T_i)) \leq \lambda(T_j)(\rho(T_j))$ and ψ_R being tree embeddings, we get that ψ is a tree embedding proving that $T_i \sqsubseteq_{\mathcal{T}} T_j$. Contradiction to the badness of $(T_i)_{i \in \mathbb{N}}$. □

Remark 2. *Kruskal's theorem is actually slightly more general: it can also handle the case where the children are ordered. In this lecture however we will only need unordered trees.*

Definition 7. Given a (non necessarily finite) qo alphabet $(Q, \leq)$, the set of words over Q is $Q^* := \{a_1 \dots a_n \mid q_i \in Q\}$. Given a word $w = a_1 \dots a_n \in Q^*$, the set of its positions is $\mathrm{pos}(w) = \{1, \dots, n\}$. A *word embedding* from $w = a_1 \dots a_n \in Q^*$ to $w' = b_1 \dots b_m \in Q^*$ is an injective function $\varphi\colon \mathrm{pos}(w) \to \mathrm{pos}(w')$ such that for all $i \in \mathrm{pos}(w)$:

1. $a_i \leq b_{\varphi(i)}$, and
2. for all $j \in \mathrm{pos}(w)$, if $i \leq j$ then $\varphi(i) \leq \varphi(j)$.

The word embedding ordering $\leq^*$ over Q^* is defined so that $w \leq^* w'$ if there is a word embedding from w to w'.

As an example, take the wqo $(\mathbb{N}, \leq)$ as the alphabet, then we have $9\,8\,3\,2 \leq^* 4\,9\,9\,0\,2\,5\,2\,7$ but $9\,8\,3 \not\leq^* 8\,9\,9\,1\,1\,1$. When the alphabet is a finite set Σ ordered by $=$, word embedding is called the *subword* ordering, written $\preccurlyeq$. For example, with $\Sigma = \{a, b, c\}$ we have: $a\,c\,b \preccurlyeq c\,a\,b\,c\,b\,a$ but $a\,c \not\preccurlyeq c\,b\,a$.

Here we can derive that words over a wqo form a wqo as a corollary of Kruskal's theorem. The lemma was however already known before Kruskal's result, thanks to a proof in a 1952 paper by Higman, after whom the lemma is named.

Corollary 2 (Higman's Lemma)**.** *Let $(Q, \leq)$ be a wqo. Then $(Q^*, \leq^*)$ is a wqo.*

Proof. A word $a_1, \dots, a_n \in Q^*$ is a Q-labelled tree: a_1 is the label of the root with single child labelled by a_2, with single child labelled by a_3, and so on. Subword ordering is then an instance of tree embedding, thus by Theorem 2 and Lemma 1 we get the result. □

Note, by contrast, that the prefix ordering and the lexicographic ordering on words are both not a wqo, not even for finite alphabets.