Sebastian Muskalla
Prof. Dr. Roland Meyer
TU Braunschweig

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Out: April 26 Due: May 2

You can submit your solutions on Tuesday, May 2, at the beginning of the exercise classes (since Monday, May 1, is a public holiday.) I will grade and return the submissions as soon as possible!

Assume that we are in the same setting as in Section 6, Büchi and CoBüchi games: $G = (V_\square \cup V_\bigcirc, R)$ is a finite, deadlock-free game arena, and $B \subseteq V$ is a winning set. Recall the definitions of the sets B^i and P^i:

$$\begin{aligned} B^0 &= B, \\ P^i &= V \setminus \mathrm{Attr}_\bigcirc(B^i), \\ B^{i+1} &= B \setminus \mathrm{CPre}_\square(P^i). \end{aligned}$$

Recall that the controlled predecessors of a set $X \subseteq V$ were defined as

$$\mathrm{CPre}_\star(X) = \left\{x \in V_\star \;\middle|\; \exists (x,y) \in R \colon y \in X\right\} \cup \left\{x \in V_{\overline{\star}} \;\middle|\; \forall (x,y) \in R \colon y \in X\right\}$$

and that using this definition, the attractor can be defined as follows.

$$\begin{aligned} \mathrm{Attr}^0_\star(X) &= X \\ \mathrm{Attr}^{i+1}_\star(X) &= \mathrm{Attr}^i_\star(X) \cup \mathrm{CPre}_\star\left(\mathrm{Attr}^i_\star(X)\right) \end{aligned}$$

Exercise 1: Proof of Lemma 6.9

Prove Lemma 6.9 from the lecture notes:

The sets B^i form a descending, the sets W_i form an ascending chain. There is an index $m \in \mathbb{N}$ such that the chains **simultaneously** become stationary.

$$B = B^0 \supseteq B^1 \supseteq \ldots \supseteq B^m = B^{m+1} = \bigcap_{i \in \mathbb{N}} B^i$$

$$P^0 \subseteq P^1 \subseteq \ldots \subseteq P^m = P^{m+1} = \bigcup_{i \in \mathbb{N}} P^i$$

Exercise 2: A more intuitive definition of recurrence sets

In this exercise, we give a more intuitive definition of the recurrence sets B^i, and we prove that it is equivalent to the definition in the lecture.

We need a slightly modified attractor construction:

$$A^0_{☆}(X) = \varnothing$$
$$A^{i+1}_{☆}(X) = A^i_{☆}(X) \cup \mathrm{CPre}_{☆}(A^i_{☆}(X) \cup X)$$
$$\mathrm{Attr}^+_{☆}(X) = \bigcup_{i \in \mathbb{N}} A^i_{☆}(X)$$

Now we give an alternative definition for the sets B^i, here called B^i_{ex}:

$$B^0_{ex} = B$$
$$B^{i+1}_{ex} = B \cap \mathrm{Attr}^+_{\bigcirc}\left(B^i_{ex}\right)$$

a) Describe the difference between $\mathrm{Attr}^+_{☆}(B)$ and $\mathrm{Attr}_{☆}(B)$ in your own words.
(Not more than 2 sentences please.)

b) Formally prove using induction on i that $A^i_{☆}(B) \cup B = \mathrm{Attr}^i_{☆}(B)$ for all $i \in \mathbb{N}$ and conclude $\mathrm{Attr}^+_{☆}(B) \cup B = \mathrm{Attr}_{☆}(B)$.

c) Formally prove using induction on i that $B^i = B^i_{ex}$ for all $i \in \mathbb{N}$.

Hint: In the induction step, you essentially need to prove

$$V \setminus \mathrm{Attr}^+_{\bigcirc}(B^i) = \mathrm{CPre}_{\square}\left(V \setminus \mathrm{Attr}_{\bigcirc}(B^i)\right) .$$

Part b) of this exercise is crucial for proving this statement.