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Games with perfect information
Exercise sheet 9

TU Braunschweig
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Out: June 1 Due: June 16

Submit your solutions until Friday, June 16, 14:00, in the box next to office 343. Note that due to the excursion week, there is no lecture, exercise class and submission in the week of June 5 to 9.

Exercise 1: Examples for PTAs

Consider the ranked alphabet $\Sigma = \{a_{/2}, b_{/2}\}$. Note that Σ-trees are so-called *full* infinite binary trees.

a) Consider the PTA $A_1 = (\Sigma, \{q_0, q_1\}, q_0, \rightarrow, \Omega)$ with

$$\begin{aligned} \rightarrow_a &= \{(q_0, (q_1, q_1))\}, \\ \rightarrow_b &= \{(q_1, (q_0, q_0))\}, \\ \Omega(q_0) &= \Omega(q_1) = 0. \end{aligned}$$

Describe its language $\mathcal{L}(A_1)$.

b) Consider the PTA $A_2 = (\Sigma, \{q_+, q_-\}, q_+, \rightarrow, \Omega)$ with

$$\begin{aligned} \rightarrow_a &= \{(q_+, (q_-, q_-))\}, \\ \rightarrow_b &= \{(q_+, (q_+, q_-)),\ (q_+, (q_-, q_+)),\ (q_-, (q_-, q_-))\}, \\ \Omega(q_-) &= 0, \qquad \Omega(q_+) = 1. \end{aligned}$$

Formally prove that $\mathcal{L}(A_2)$ is the set of Σ-trees in which exactly one node is labeled by a.

Remark: A_2 is non-deterministic, and one can prove that there is no deterministic PTA A accepting the same language.

c) Present a PTA A_3 whose language is the set of Σ-trees in which exactly one branch contains infinitely many nodes labeled by a.

Argue that your automaton indeed has this property.

Exercise 2: Closure properties of regular languages of infinite trees

Prove that the class of regular languages of infinite trees is closed under union, intersection, and projection.

Let $A = (\Sigma, Q, q_0, \rightarrow, \Omega)$, $A' = (\Sigma, Q', q_0', \rightarrow', \Omega')$ be PTAs over the same ranked alphabet Σ.

a) Show how to construct a PTA $A_\cup$ with $\mathcal{L}(A_\cup) = \mathcal{L}(A) \cup \mathcal{L}(A')$.

b) Show how to construct a PTA $A_\cap$ with $\mathcal{L}(A_\cap) = \mathcal{L}(A) \cap \mathcal{L}(A')$.

   Hint: Use Rabin's tree theorem.

c) Let Σ' be a ranked alphabet, and $f \colon \Sigma \rightarrow \Sigma'$ be a **rank preserving function**, i.e. we have $\mathrm{rank}_\Sigma(a) = \mathrm{rank}_{\Sigma'}(f(a))$ for all $a \in \Sigma$. For a Σ-tree $\mathcal{T}$, we define $f(\mathcal{T})$ to be the Σ'-tree in which the label a of each node is replaced by $f(a)$. Note that the fact that f is rank-preserving is crucial for $f(\mathcal{T})$ being a Σ'-tree.

   For a language of Σ-trees $\mathcal{L}$, we define

$$f(\mathcal{L}) = \{ f(\mathcal{T}) \mid \mathcal{T} \in \mathcal{L} \} .$$

   Show how to construct a PTA $A_f = (\Sigma', Q_f, q_{0f}, \rightarrow_f, \Omega_f)$ with $\mathcal{L}(A_f) = f(\mathcal{L}(A))$.

Exercise 3: Proof of Lemma 9.13

Let $\mathcal{T}$ be a Σ-tree and let A be a PTA. Consider the parity game $\mathcal{G}(\mathcal{T}, A)$ as defined in Definition 9.12 from the lecture notes.

a) The game arena of $\mathcal{G}(\mathcal{T}, A)$ is not necessarily deadlock-free.

   In which case can deadlocks occur?

   Modify the game arena such that it becomes deadlock free, while preserving Lemma 9.13 from the lecture notes.

   How can one modify the automaton without changing its language such that $\mathcal{G}(\mathcal{T}, A)$ is deadlock-free without modification?

b) Assume that refuter has a positional winning strategy $s_\bigcirc$ from position (ε, q_0) in $\mathcal{G}(\mathcal{T}, A)$.

   Present an accepting run of $\mathcal{T}$ on A.

   Hint: Construct the run inductively, guided by $s_\bigcirc$.

c) Assume that prover has a positional winning strategy $s_\square$ from position (ε, q_0) in $\mathcal{G}(\mathcal{T}, A)$.

   For each run of A on $\mathcal{T}$, identify a branch on which the acceptance condition is violated.