Games with perfect information **Exercise sheet 2**	TU Braunschweig
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Out: April 11 Due: April 18

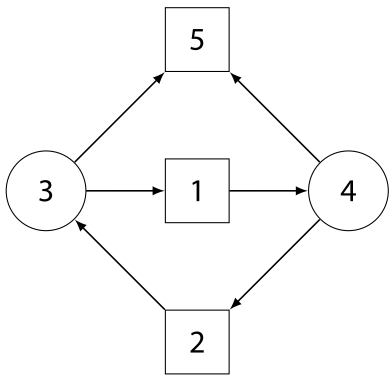
Submit your solutions on Wednesday, April 18, during the lecture.
Please submit in groups of three persons.

Exercise 1: Tic-tac-toe
Consider the popular game **tic-tac-toe**,
see e.g. `https://en.wikipedia.org/wiki/Tic-tac-toe`.

Formalize the game, i.e. formally define a game $\mathcal{G} = (G, win)$ consisting of a game arena and a winning condition that imitates the behavior of tic-tac-toe.

Assume that player $\bigcirc$ makes the first mark, and the other player wins in the case of a draw.

Exercise 2: Positional and uniform strategies
If a game arena has finitely many positions, we can explicitly give it as a graph. For this exercise, we consider a game on the following game arena $G = (V, R)$. Positions owned by the universal player $\square$ are drawn as boxes, positions owned by the existantial player $\bigcirc$ as circles. The numbers should denote the names of the vertices, i.e. $V = \{1, \ldots, 5\}$.



We consider the following winning condition: A maximal play is won by the existential player if and only if the positions 3, 4 and 5 are each visited exactly once.

a) What is the winning region for each of the players?

   Present a single strategy $s_{\bigcirc}\colon Plays_{\bigcirc} \to V$ that is winning from all positions x in the winning region $W_{\bigcirc}$ of the existential player. Argue shortly why your strategy is indeed winning from these positions.

   Note: Such a strategy is called a *uniform* winning strategy.

b) For each vertex $x \in W_{\bigcirc}$ in the winning region of the existential player, present a positional strategy for existential player $s_{\bigcirc,x}\colon \{3, 4\} \to R$ such that $s_{\bigcirc,x}$ is winning from x.

c) Prove that there is no uniform positional winning strategy for the existential player, i.e. no single positional strategy that wins from all $x \in W_{\bigcirc}$.

d) Consider the modified graph that is obtained by adding a vertex 6 owned by $\bigcirc$ and the arcs $(6, 3)$ and $(6, 4)$.
Prove that position 6 is winning for the existential player, but there is no positional winning strategy from 6.

Exercise 3: Multiplayer games

Assume that three-player games are defined analogously to two-player games, i.e. they are played on a directed graph with an ownership function $owner\colon V \to \{1, 2, 3\}$, and their winning condition is a function $win\colon Plays_{max} \to \{1, 2, 3\}$. (Winning) strategies are defined similar to two-player games.

For every three-player game $\mathcal{G}_{3p} = (G_{3p}, win_{3p})$, where $G_{3p} = (V_1 \uplus V_2 \uplus V_3, R)$ and each player $i \in \{1, \ldots, 3\}$, show how to construct a two-player game $\mathcal{G}_i = (G_i, win_i)$ with $G_i = (V_{\Box} \uplus V_{\bigcirc}, R)$ such that:

- The underlying directed graph is the same, i.e. $V_1 \uplus V_2 \uplus V_3 = V_{\Box} \uplus V_{\bigcirc}$.
- Each node $x \in V_1 \uplus V_2 \uplus V_3$ is winning for player i in the game $\mathcal{G}_{3p}$ if and only if it is winning for player $\bigcirc$ in the game $\mathcal{G}_i$.

Prove that your constructed game $\mathcal{G}_2$ has the desired properties.

Exercise 4: Deadlocks

Many books in the literate only consider games that are deadlock-free, meaning every position $x \in V$ has at least one outgoing arc $(x, y) \in R$ (where self-loops, i.e. $x = y$, are allowed).

Assume that $\mathcal{G} = (G, win)$ is a game that may contain deadlocks. Furthermore, we assume that the winning condition has the property that any finite play ending in a deadlock is lost by the player owning the last position.

Construct a game $\mathcal{G}' = (G', win')$ that does not contain deadlocks. The new game arena G' should be obtained from G by adding vertices and arcs, in particular each position of the old game is a position of the new game, $V \subseteq V'$.

Your construction should guarantee that each position $x \in V$ of the old game is winning in the new game for the same player for which it was winning in the old game. Argue why it has this property.