Theory Solvers for Linear Arithmetic and Equality Logic with Uninterpreted Functions

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Predicate Logic . . .

. . . is an elegant way to express what we mean:

$$\exists x_{\text{state}}, x'_{\text{state}} : \text{init}(x_{\text{state}}) \wedge \text{bad}(x'_{\text{state}}) \wedge \text{reach}(x_{\text{state}}, x'_{\text{state}})$$

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Idea: Restrict to certain structures

Introducing Theories

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- *T-implication*: Implication restricted to relevant structures ($A \models_T B$)

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The Satisfiability Modulo Theory Problem for a theory T: Given a formula A is it satisfied by a model that is allowed by the theory T?

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- (boolean) satisfiable $\rightarrow$ iff there is a model, a corresponding boolean assignment will be found

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- Cycle: Exactly 0 steps from x to x; constraints < 0 cannot be satisfied

Example for Difference Arithmetic

Which of the following conjunctions is satisfiable?

$$A \equiv (y - x \leq 2) \wedge (z - y \leq -3) \wedge (x - z \leq \quad 7)$$
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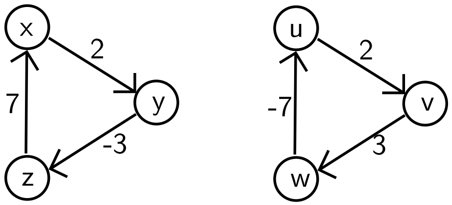

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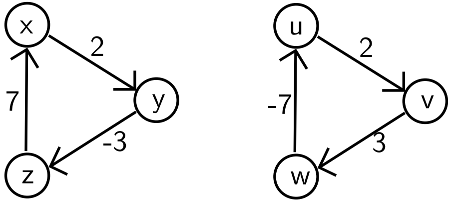

Left graph: Weight of 6 $\rightarrow$ satisfiable e.g. not $(6, 6, 6)$ but $(7, 3, 0)$
Right graph: Weight of -2 $\rightarrow$ unsat.: u-to-u takes 0 steps but only -7 allowed

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- $x - y - 1 \leq 1$
- $x - y \leq 2$ (isolate constants)
- $x - y - s = 0$ (s is a fresh variable) $s \leq 2$

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Basic idea: Adjust variables until they fit.

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- How? Swapping variables with and without bounds (pivoting)

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Trivial once only one variable left

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- No recursive call finds a solution? Unsatisfiable.

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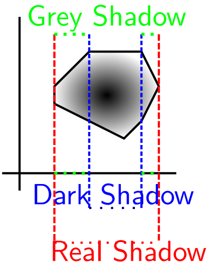
Adding constraints: The constraints describe where we search (recursively)



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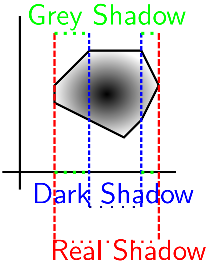

- **Real Shadow:** Eliminate variable as before

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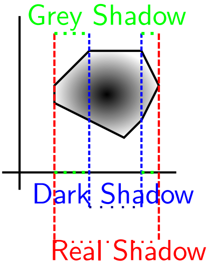

- **Dark Shadow:** Constraints enforce "big" gaps for the eliminated variable

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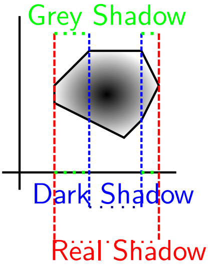

- **Grey Shadow:** Real Shadow without dark shadow

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- **Grey Shadow:** Excluding the dark shadow from the grey shadow yields a finite set of possible constraints. Try them all.

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Transitive closure for equal-edges

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Connected components get the same value

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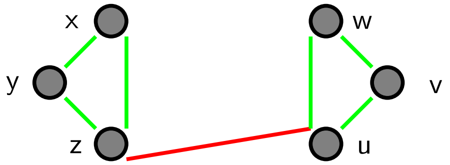
What if we add $(x = w)$?



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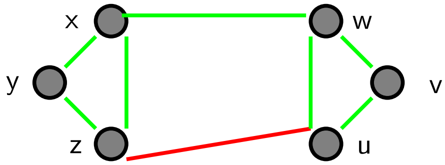
What if we add $(x = w)$?



Equality Logic with Uninterpreted Functions

We want to add functions.
First try: Introduce variables for each term:

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Example:

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Thank you for your Attention