SS 2013 June 26th, 2013

Exercises to the lecture Logics
Sheet 6

Jun.-Prof. Dr. Roland Meyer Due July 5th, 2013, 12:00pm

Exercise 6.1 [Non-standard models]

Let $S = (F, P)$ be the signature with function symbols $F = \{\mathsf{0}/0, \mathsf{1}/0, +/2, */2\}$ and predicate symbols $P = \{\leqslant/2\}$. Furthermore, let $\mathcal{N} = (\mathbb{N}, I_{\mathbb{N}})$ be the S-structure, in which the domain consists of the natural numbers and the symbole $\mathsf{0}$, $\mathsf{1}$, $+$, $\leqslant$, and $*$ are interpreted as usual. Finally, let $\mathcal{T}_{\mathcal{N}}$ be the set of all closed formulae that are satisfied by $\mathcal{N}$.

a) Consider the set
$$\mathcal{T}'_{\mathcal{N}} = \mathcal{T}_{\mathcal{N}} \cup \{\underbrace{\mathsf{1} + \cdots + \mathsf{1}}_{n} \leqslant \mathsf{x} \mid n \geqslant 1\},$$
in which x is a variable. Show that $\mathcal{T}'_{\mathcal{N}}$ is satisfiable. *Hint:* Employ the Compactness Theorem.

b) Let $\mathcal{M}$ and $\mathcal{M}'$ be structures over the same signature S'. The structures $\mathcal{M}$ and $\mathcal{M}'$ are called *elementarily equivalent* if for every closed formula A in predicate logic over S, we have: $\mathcal{M} \models A$ if and only if $\mathcal{M}' \models A$. Show that every structure $\mathcal{M}$ that satisfies $\mathcal{T}'_{\mathcal{N}}$ is elementarily equivalent to $\mathcal{N}$.

c) If $\mathcal{M} = (D, I)$ and $\mathcal{M}' = (D', I')$ are structures over the same signature, we call $\mathcal{M}$ and $\mathcal{M}'$ *isomorphic* if there is a bijection $\varphi : D \to D'$ with
$$p^{\mathcal{M}}(d_1, \ldots, d_k) = p^{\mathcal{M}'}(\varphi(d_1), \ldots, \varphi(d_k)) \qquad \text{for all } d_1, \ldots, d_k \in D \text{ and}$$
$$\varphi(f^{\mathcal{M}}(d_1, \ldots, d_\ell)) = f^{\mathcal{M}'}(\varphi(d_1), \ldots, \varphi(d_\ell)) \qquad \text{for all } d_1, \ldots, d_\ell \in D$$
for every k-ary predicate symbol p and every ℓ-ary function symbol f. Conclude from a) and b) that there is a structure that die elementarily equivalent but not isomorphic to $\mathcal{N}$.

Exercise 6.2 [Satisfiability and deducibility]

Show that the following problems are semi-decidable but not decidable:

a) ~~Given a formula A in predicate logic, decide whether A is satisfiable.~~

b) Given two formulae A and B in predicate logic, decide whether $A \models\!\dashv B$.

Exercise 6.3 [Undecidability]

A context-free grammar is called *linear* if in each rule, the right-hand side contains at most one occurrence of a nonterminal symbol. Show that the following problem is undecidable: Given linear context-free grammars G_1 and G_2, is the set $L(G_1) \cap L(G_2)$ empty?

Exercise 6.4 [Reductions]

Let $\mathcal{C}$ be a class of problems. A problem A is called $\mathcal{C}$-*hard* if every problem in $\mathcal{C}$ is many-one-reducible to A. Prove: If A is $\mathcal{C}$-hard and A many-one-reducible to a problem B, then B is $\mathcal{C}$-hard as well.

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