Summaries for Context-Free Games

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Language-Theoretic Verification

Verification

Verification problem:

Given: Source code of program P and specification φ .

Question: Does runtime behavior of P satisfy φ ?

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Question: Does runtime behavior of P satisfy φ ?

Language-theoretic approach:

$\mathcal{L}_P =$ possible program executions

$\mathcal{L}_\varphi =$ valid executions

Decide: $\mathcal{L}_P \subseteq \mathcal{L}_\varphi$

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Good: $\mathcal{L}_\varphi$ usually easy (regular)

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Good: $\mathcal{L}_\varphi$ usually easy (regular)

Bad: $\mathcal{L}_P$ usually not even context free

↳ Problem is undecidable

↳ Need to approximate $\mathcal{L}_P$

Language-theoretic verification

Semantics:

$$\mathcal{L}_P \;=\; \mathcal{L}_{CF} \cap \mathcal{L}_{Data}$$

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$\mathcal{L}_{Data}$ is anything: *Var* is infinite and $\mathcal{L}_x$ is arbitrary

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Lessons in life:

Handle control flow using techniques from automata theory

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CEGAR loop [Podelski et al. since 2010]

Counterexample-guided abstraction refinement

Init $\mathcal{L}_S := \mathcal{L}_\varphi$

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$$\text{Init } \mathcal{L}_S := \mathcal{L}_\varphi$$

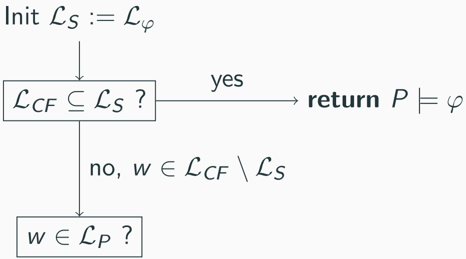
$$\downarrow$$

$$\boxed{\mathcal{L}_{CF} \subseteq \mathcal{L}_S \text{ ?}}$$

Counterexample-guided abstraction refinement

$$\begin{array}{ccc} \text{Init } \mathcal{L}_S := \mathcal{L}_\varphi & & \\ \downarrow & & \\ \boxed{\mathcal{L}_{CF} \subseteq \mathcal{L}_S \text{ ?}} & \xrightarrow{\text{yes}} & \textbf{return } P \models \varphi \end{array}$$

Counterexample-guided abstraction refinement



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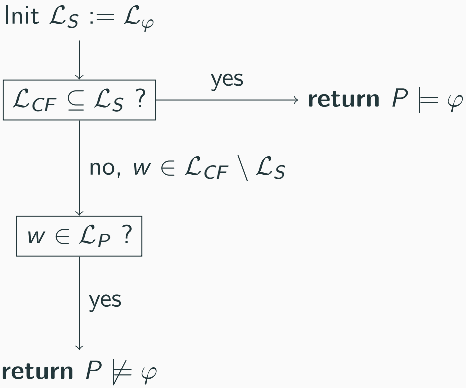

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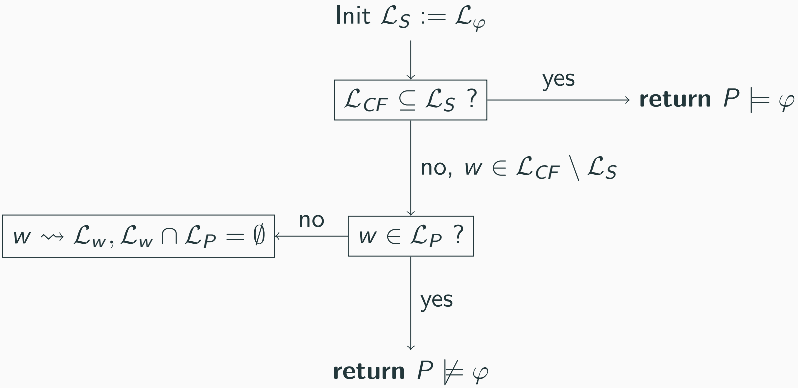

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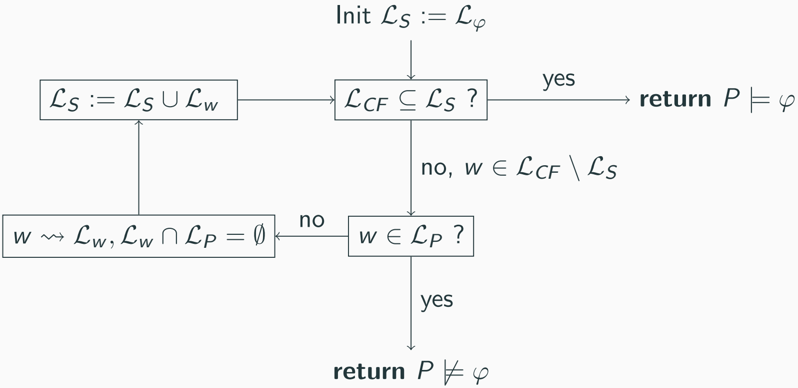

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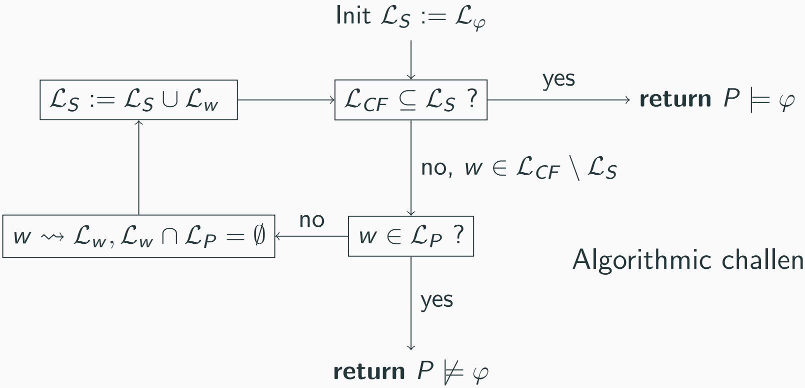

Algorithmic challenges

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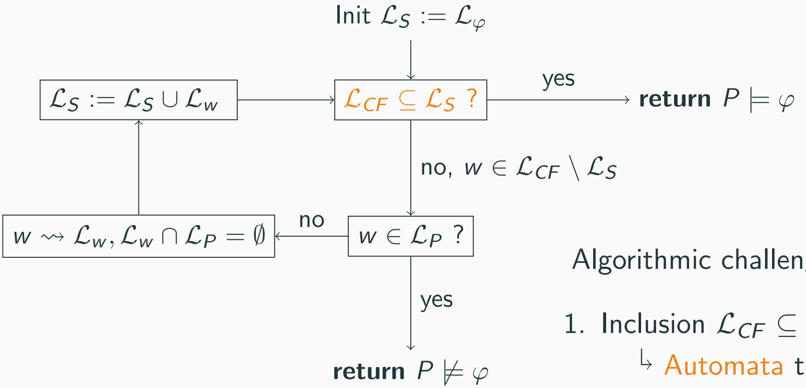

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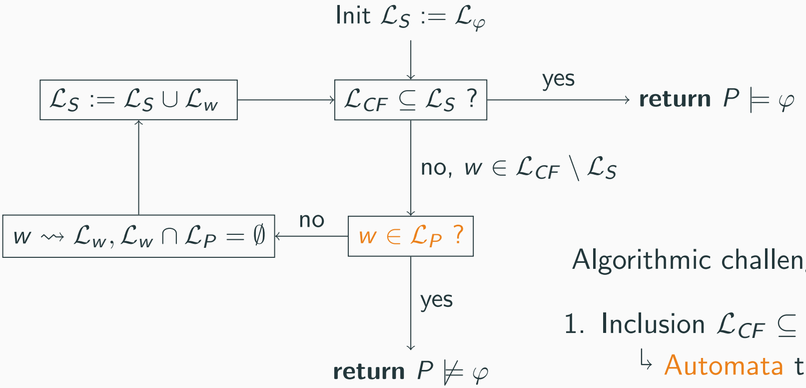

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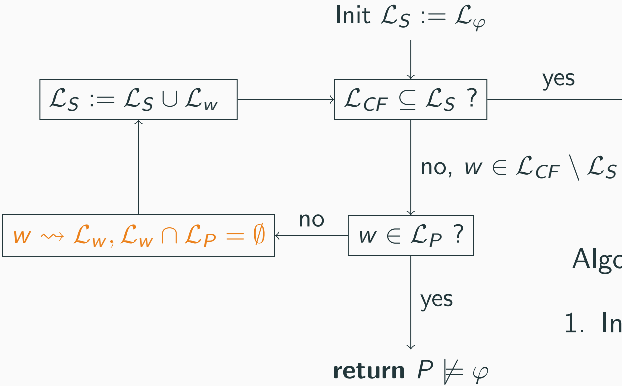

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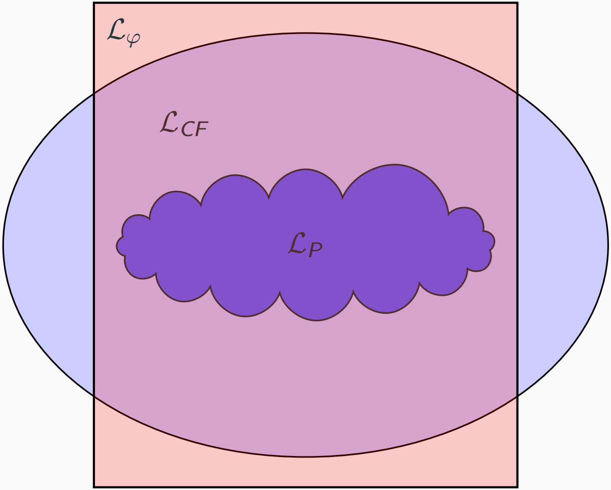

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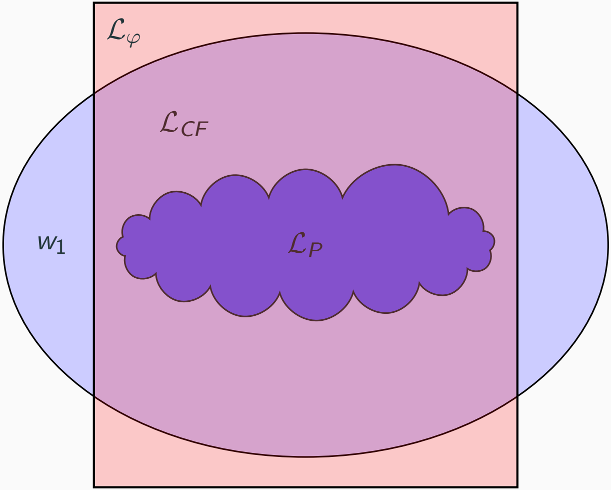

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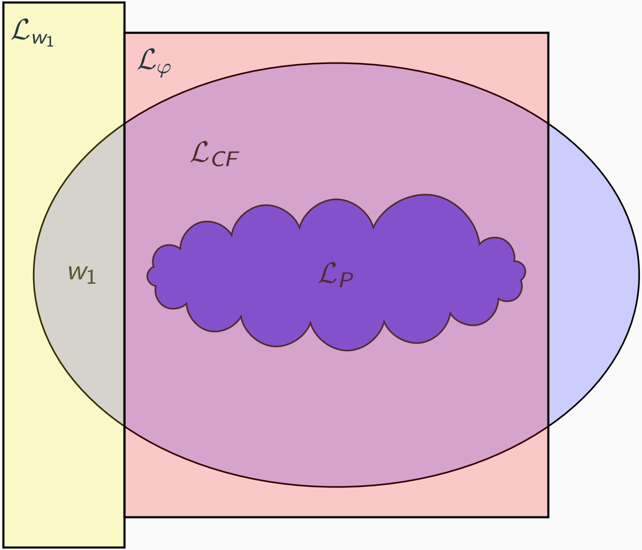

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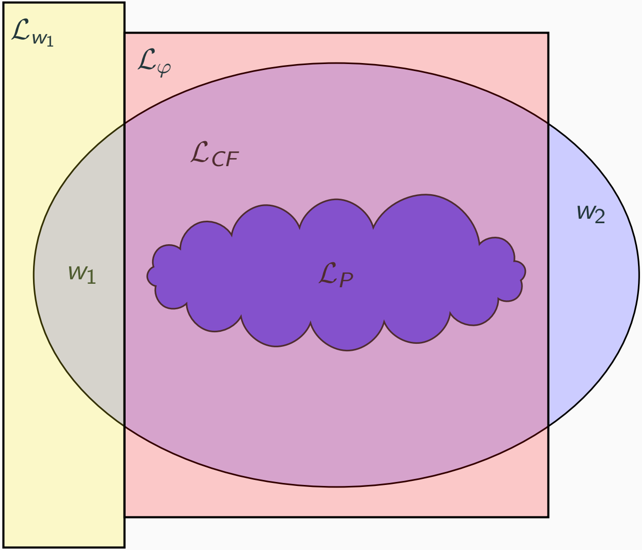

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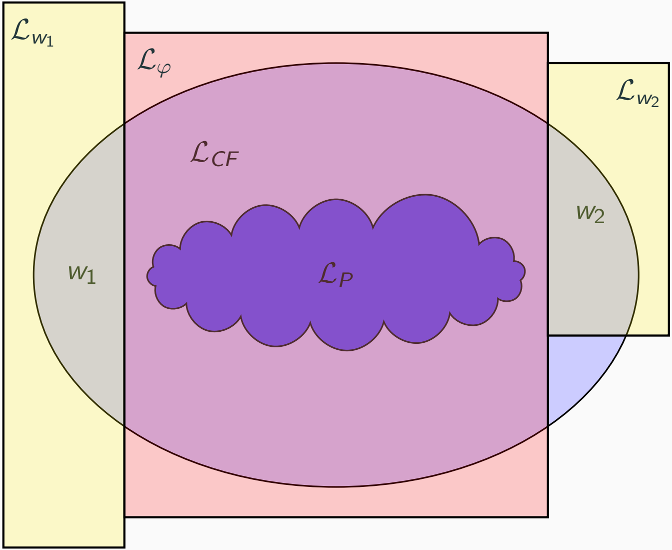

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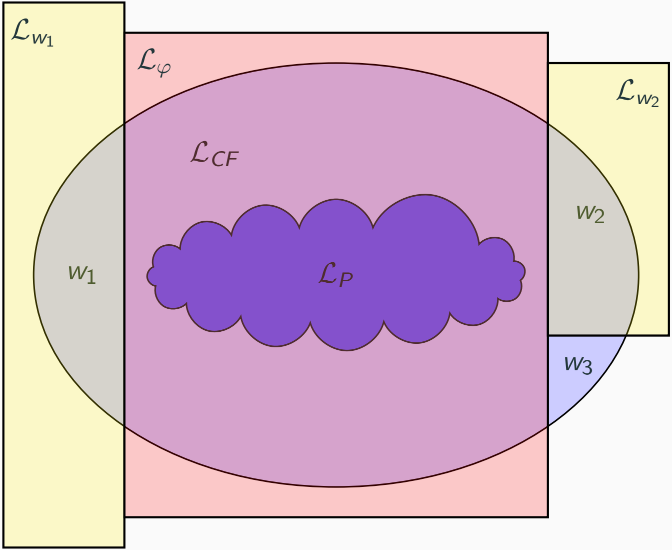

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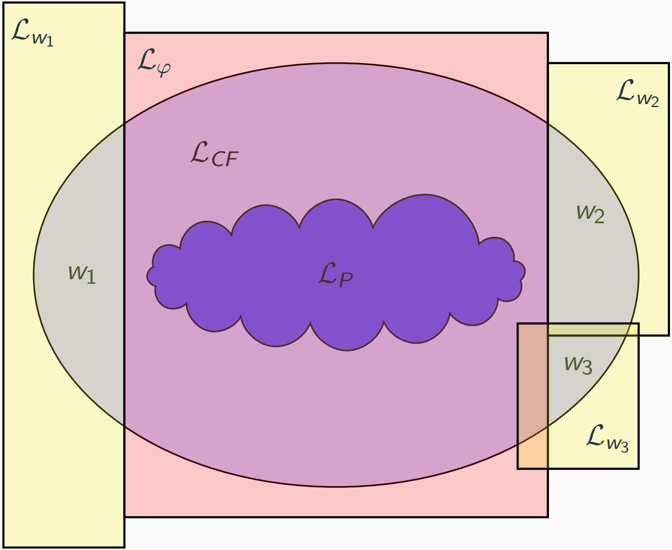
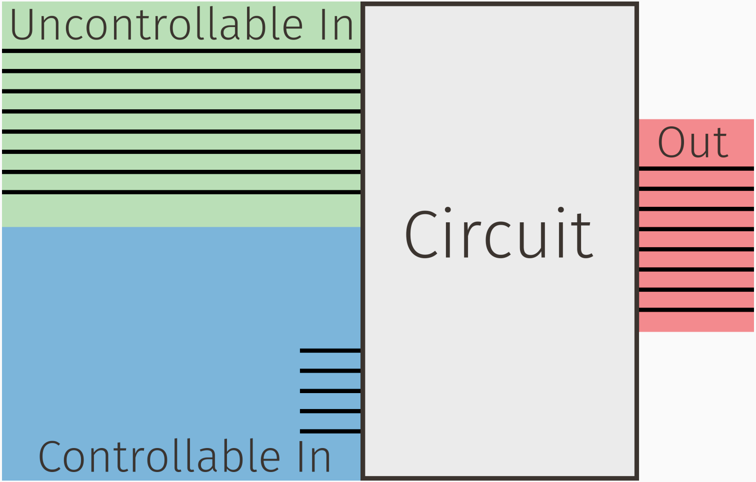


Language-Theoretic Synthesis

Synthesis



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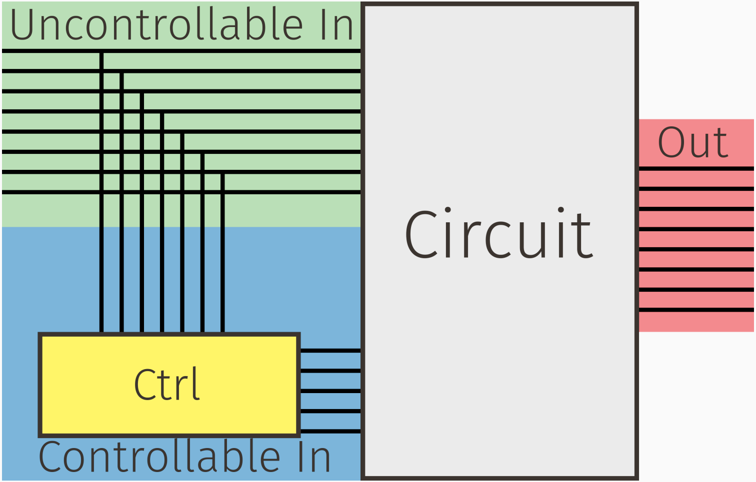

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Given: Program template T and specification φ .

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Approach:

Language-theoretic synthesis

CEGAR loop

Language-theoretic synthesis

Model the control flow of a template as a grammar

Two types of non-determinism

Language-theoretic synthesis

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Two types of non-determinism

Demonic / Uncontrollable
non-determinism

```
proc F()
    if (x == 0)
        G()
    else
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$$F \rightarrow \texttt{read(x,0)}\,G \mid \texttt{read(x,1)}\,H$$

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proc F()
    if ???
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```

$$F \rightarrow G \mid H$$

Language-theoretic synthesis

Algorithmically:

Model as a (context-free) two player perfect information game

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Player ○ represents uncontrollable non-determinism

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Is there a strategy s for player $\Box$ to resolve the controllable non-determinism so that

$$\mathcal{L}(G@s) \subseteq \mathcal{L}(A) \ ?$$

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Player □ represents controllable non-determinism

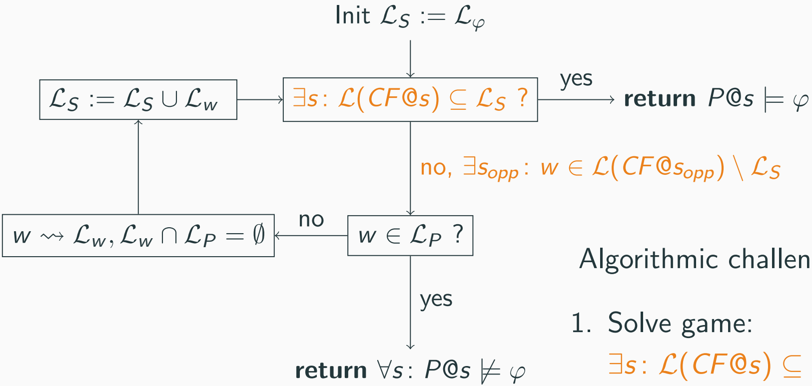
Is there a strategy s for player □ to resolve the controllable non-determinism so that

$$\mathcal{L}(G@s) \subseteq \mathcal{L}(A) \text{ ?}$$

From language-theoretic verification to synthesis:

Replace the inclusion check $\mathcal{L}(G) \subseteq \mathcal{L}(A)$ in the CEGAR loop by a strategy synthesis

Language-theoretic synthesis



Algorithmic challenges

1. Solve game:
   $\exists s\colon \mathcal{L}(CF@s) \subseteq \mathcal{L}_S$?
2. Membership $w \in \mathcal{L}_P$
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Context-Free Games

Context-free games - Input

Input:

Context-free grammar with ownership partitioning of the non-terminals

$$\begin{aligned} X_{\bigcirc} &\rightarrow \quad aY \quad | \quad \varepsilon \\ Y_{\square} &\rightarrow \quad bX \end{aligned}$$

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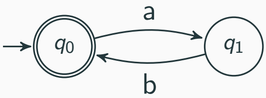
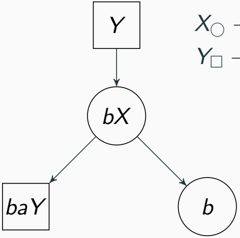
Finite automaton over terminals T_G



Context-free games - Game arena

Game arena:



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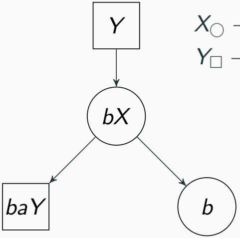

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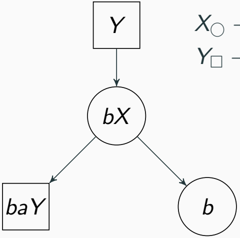

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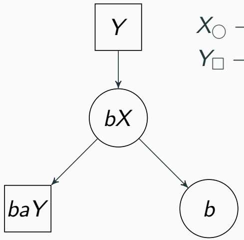

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Ownership: Owner of $wX\gamma$ is the owner of X

Context-free games - Winning conditions

Winning conditions:

Inclusion game:

Derive a terminal word $w \in \mathcal{L}(A)$ or infinite derivation

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Here:

Consider inclusion game for player prover $\square$

Consider non-inclusion game for player refuter $\bigcirc$

Context-free games - Algorithms

State-of-the-art in verification:

Context-free games - Algorithms

State-of-the-art in verification:

Saturation

Compute state space of a pushdown

Stack content represented as a regular language

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- Compute effect of function calls as input output relation
- Stack content not represented
- Used more often in SVComp

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Summaries for context-free games

How to decide which player wins the game?

Fixed-point iteration over a suitable summary domain

Now:

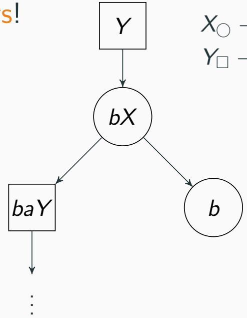
1. Explain & define domain
2. Explain fixed-point iteration

Formulas over the Transition Monoid

The tree of plays

How to decide whether refuter can win from a given position?

Consider the tree of plays!

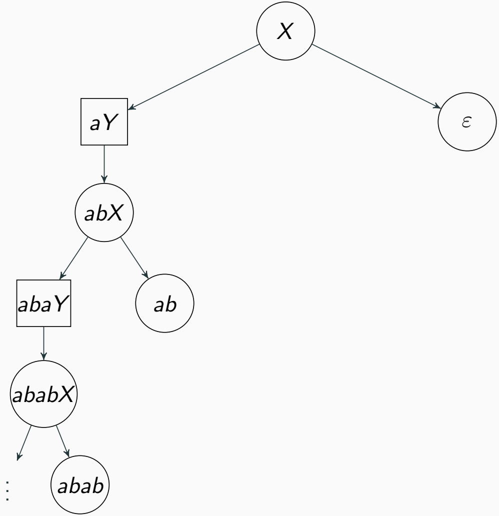


$X_{\bigcirc} \to aY \mid \varepsilon$
$Y_{\Box} \to bX$

Refuter wins non-inclusion in $(ab)^*$ by picking $X \to \varepsilon$

Y is a winning position for refuter $\bigcirc$

The tree of plays - Example



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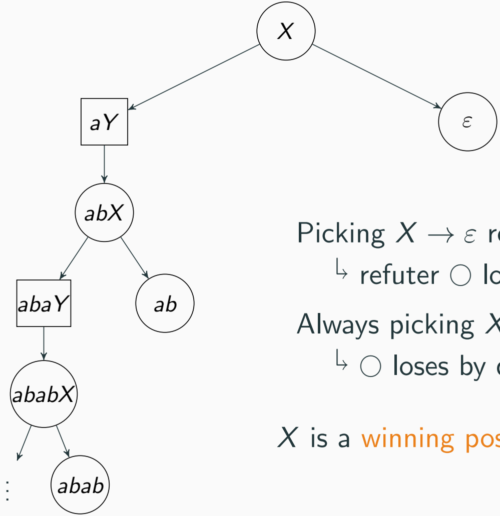

$X_{\bigcirc} \to aY \mid \varepsilon$
$Y_{\Box} \to bX$

Picking $X \to \varepsilon$ results in word in $(ab)^*$

↳ refuter $\bigcirc$ loses non-inclusion

Always picking $X \to aY$ results in infinite play

↳ $\bigcirc$ loses by definition

X is a winning position for prover $\Box$

Formulas

Problem:

Tree is usually infinite

Formulas

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Observation 1:

Labels of inner nodes do not matter for inclusion

Formulas

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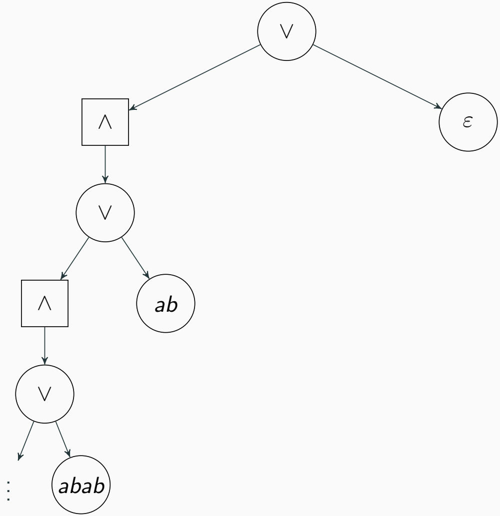
Understand tree as (infinite) positive Boolean formula over words

Formulas - Example



Formulas

Remaining problems:

1. Formulas are *still* infinite
2. Even the set of atomic propositions ${T_G}^*$ is infinite

↳ Tackle 2. first

Equivalence relation

Observation 2:

The words are not important — only the state changes matter

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$$\begin{aligned} & w \sim_A v \\ \text{iff} \quad & \forall q, q' \in Q : \end{aligned}$$

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Define equivalence relation $\sim_A$ such that
words are equivalent iff they induce the same state changes on A

$$\begin{array}{ll} & w \sim_A v \\ \text{iff} & \forall q, q' \in Q : \quad q \xrightarrow{w} q' \quad \text{iff} \quad q \xrightarrow{v} q' \end{array}$$

Equivalence relation

Observation 2:

The words are not important — only the state changes matter

Define equivalence relation $\sim_A$ such that
words are equivalent iff they induce the same state changes on A

$$\begin{aligned} & w \sim_A v \\ \text{iff} \quad & \forall q, q' \in Q : \quad q \xrightarrow{w} q' \quad \text{iff} \quad q \xrightarrow{v} q' \end{aligned}$$

M_A is the set of all equivalence classes $[w]$ of $\sim_A$

${T_G}^*$ is partitioned into equivalence classes of $\sim_A$

Transition monoid

Represent equivalence classes by boxes:

$$\mathrm{box}(w) = \left\{ (q, q') \in Q \times Q \;\middle|\; q \xrightarrow{w} q' \right\} \in \mathcal{P}(Q \times Q)$$

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Boxes correspond to procedure summaries for programs
(in a precise sense)

Transition monoid - Example

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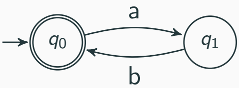
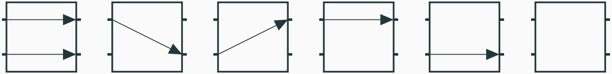


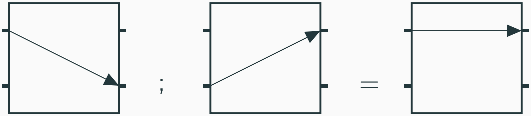


$id = [\varepsilon]$ $\quad [a] \quad [b] \quad [ab] \quad [ba] \quad [aa] = [bb]$

All other boxes represent empty equivalence classes

Relational composition of boxes

Boxes can be composed using relational composition ;



$[a]$ $[b]$ $[ab]$

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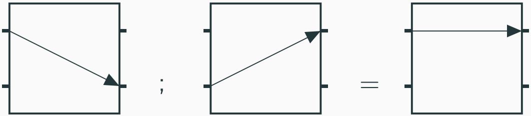

$[a]$ $[b]$ $[ab]$

Monoids are isomorphic:

$$\big(M_A,\ .\ ,[\varepsilon]\big) \;\cong\; \big(\underbrace{\mathsf{box}(T_G{}^*)}_{\subseteq \mathcal{P}(Q\times Q)},\ ;\ ,\mathsf{box}(\varepsilon)\big)$$

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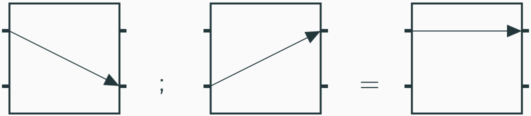

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Monoids are isomorphic:

$$\big(M_A,\ .\ ,[\varepsilon]\big) \ \cong\ \big(\underbrace{\mathrm{box}(T_G{}^*)}_{\subseteq \mathcal{P}(Q\times Q)},\ ;\ ,\mathrm{box}(\varepsilon)\big)$$

↳ Up to $|M_A| \leq 2^{|Q|^2}$ equivalence classes

Back to games

Previously: (Infinite) positive Boolean formulas over words

Back to games

Previously: (Infinite) positive Boolean formulas over words

Now: (Infinite) positive Boolean formulas over M_A

Back to games

Previously: (Infinite) positive Boolean formulas over words

Now: (Infinite) positive Boolean formulas over M_A

Down to finitely many atomic propositions

Back to games

Previously: (Infinite) positive Boolean formulas over words

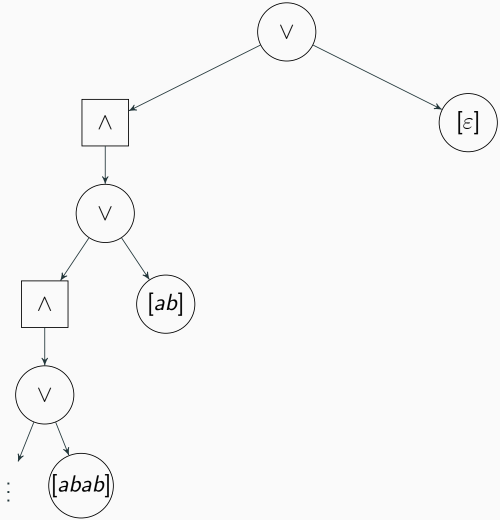
Now: (Infinite) positive Boolean formulas over M_A

Down to finitely many atomic propositions

Remaining problem:

Formulas themselves are infinite

Formulas - Example



From infinite to finite formulas

Observation 3:

Every infinite formula over M_A is logically equivalent (under suitable evaluation semantics) to some finite formula

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Restrict to finite positive Boolean formulas over M_A

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Restrict to finite positive Boolean formulas over M_A

Domain:

Finite positive Boolean formulas over M_A (up to $\Leftrightarrow$)

Least element: *false*

Partial order: Implication $\Rightarrow$

From infinite to finite formulas

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Infinite formulas define functions $F : 2^{M_A} \to \{0, 1\}$
All such functions can be represented by finite formulas

Restrict to finite positive Boolean formulas over M_A

In the example:

Infinite formula: $[\varepsilon] \vee \big([ab] \vee ([abab] \vee \ldots)\big)$
Note: $[ab] = [abab] = [ababab] = \ldots$
Finite formula: $[\varepsilon] \vee [ab]$

From infinite to finite formulas

Observation 3:

Every infinite formula over M_A is logically equivalent (under suitable evaluation semantics) to some finite formula

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Note: $[ab] = [abab] = [ababab] = \ldots$
Finite formula: $[\varepsilon] \vee [ab]$

How to compute these finite formulas in general?

Fixed-Point Iteration

Fixed point iteration

Problem:

How to compute the formulas?

Fixed-point iteration:

Translate the grammar into a system of equations

Solve using Kleene iteration

Fixed-point iteration - Example

Grammar

$$
\begin{aligned}
X_{\bigcirc} \rightarrow \quad & aY \qquad \mid \quad \varepsilon \\
Y_{\Box} \rightarrow \quad & bX
\end{aligned}
$$

System of equations

$$
\begin{aligned}
F_X = \quad & [a]; F_Y \quad \vee \quad [\varepsilon] \\
F_Y = \quad & [b]; F_X
\end{aligned}
$$

Fixed-point iteration - Example

Iteration:

Nr.	F_X	F_Y

Grammar

$$
\begin{aligned}
X_{\bigcirc} \rightarrow \quad & aY \qquad \mid \quad \varepsilon \\
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$X_{\bigcirc} \rightarrow \quad aY \quad | \quad \varepsilon$

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$F_X = \quad [a]; F_Y \quad \vee \quad [\varepsilon]$

$F_Y = \quad [b]; F_X$

Iteration:

Nr.	F_X	F_Y
0	*false*	*false*

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\begin{array}{ll}
F_X = & [a]; F_Y \;\; \vee \;\; [\varepsilon] \\
F_Y = & [b]; F_X
\end{array}
$$

Iteration:

Nr.	F_X	F_Y
0	*false*	*false*
1	$[\varepsilon]$	*false*

Fixed-point iteration - Example

Iteration:

Nr.	F_X	F_Y
0	*false*	*false*
1	$[\varepsilon]$	*false*
2	$[\varepsilon]$	$[b] = [b]; [\varepsilon]$

Grammar

$$\begin{array}{lll} X_{\bigcirc} \rightarrow & aY & \mid \quad \varepsilon \\ Y_{\square} \rightarrow & bX & \end{array}$$

System of equations

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System of equations

$$F_X = \quad [a]; F_Y \quad \vee \quad [\varepsilon]$$
$$F_Y = \quad [b]; F_X$$

Iteration:

Nr.	F_X	F_Y
0	*false*	*false*
1	$[\varepsilon]$	*false*
2	$[\varepsilon]$	$[b] = [b]; [\varepsilon]$
3	$[ab] \vee [\varepsilon]$	$[b]$

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Iteration:

Nr.	F_X	F_Y
0	*false*	*false*
1	$[\varepsilon]$	*false*
2	$[\varepsilon]$	$[b] = [b]; [\varepsilon]$
3	$[ab] \vee [\varepsilon]$	$[b]$
4	$[ab] \vee [\varepsilon]$	$[b]; ([ab] \vee [\varepsilon])$

Fixed-point iteration - Example

Grammar

$X_{\bigcirc} \to \quad aY \qquad | \quad \varepsilon$

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$F_X = \quad [a]; F_Y \quad \vee \quad [\varepsilon]$

$F_Y = \quad [b]; F_X$

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Fixed-point iteration - Example

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$$X_{\bigcirc} \to \quad aY \qquad \mid \quad \varepsilon$$
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System of equations

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$$F_Y = \quad [b]; F_X$$

Iteration:

Nr.	F_X	F_Y
0	*false*	*false*
1	$[\varepsilon]$	*false*
2	$[\varepsilon]$	$[b] = [b]; [\varepsilon]$
3	$[ab] \vee [\varepsilon]$	$[b]$
4	$[ab] \vee [\varepsilon]$	$[b]; ([ab] \vee [\varepsilon])$ $= [bab] \vee [b]$ $\Leftrightarrow [b]$

Winning Regions

Rejecting

Define the evaluation φ by

$$\begin{array}{rcl}\varphi: \quad M_A & \to & \{0,1\} \\ [w] & \mapsto & \begin{cases} 1 & (q_0, q_f) \notin \mathrm{box}(w) \quad \text{for all } q_f \in Q_f \\ 0 & \text{else} \end{cases}\end{array}$$

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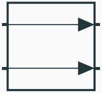
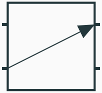
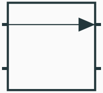


$\varphi([\varepsilon]) = 0$



$\varphi([b]) = 1$



$\varphi([ab]) = 0$

Rejecting

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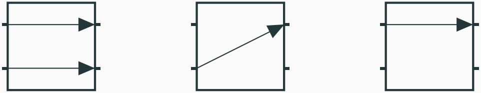

$\varphi([\varepsilon]) = 0$ $\qquad$ $\varphi([b]) = 1$ $\qquad$ $\varphi([ab]) = 0$

Sentential form $\alpha \in \vartheta$ is called rejecting if $\varphi(F_\alpha) = 1$

Winning region of prover

Theorem

The set of non-rejecting positions

$$W^{\subseteq} \;=\; \{\alpha \in \vartheta \mid \varphi(F_\alpha) = 0\}$$

is the winning region of prover $\Box$ for the inclusion game.

Winning region of prover

Theorem

The set of non-rejecting positions

$$W^{\subseteq} \;=\; \{\alpha \in \vartheta \mid \varphi(F_\alpha) = 0\}$$

is the winning region of prover $\square$ for the inclusion game.

Proof

Position $w \in \overline{\mathcal{L}(A)}$ has formula $F_w = [w]$ with $\varphi([w]) = 1$

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Show: If the current position is non-rejecting and it is the turn of

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(2) Refuter: All moves go to non-rejecting positions.

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Since the inclusion game is a safety game, staying in $W^{\subseteq}$ suffices.

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The set of non-rejecting positions

$$W^{\subseteq} \;=\; \{\alpha \in \vartheta \mid \varphi(F_\alpha) = 0\}$$

is the winning region of prover $\Box$ for the inclusion game.

In the example, starting from X:

Both $[ab]$, $[\varepsilon]$ contain (q_0, q_0)

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In the example, starting from X:

Both $[ab]$, $[\varepsilon]$ contain (q_0, q_0)

↳ $\varphi([ab]) = 0, \varphi([\varepsilon]) = 0$

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↳ $\varphi(F_X) = \varphi([ab] \vee [\varepsilon]) = 0$

Winning region of prover

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↳ X is non-rejecting

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In the example, starting from X:

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↳ $\varphi(F_X) = \varphi([ab] \vee [\varepsilon]) = 0$

↳ X is non-rejecting

Indeed, prover wins inclusion from X

Winning region of refuter

Theorem

The set of rejecting positions

$$W^{\not\subseteq} \;=\; \{\alpha \in \vartheta \mid \varphi(F_\alpha) = 1\}$$

is the winning region of refuter ○ for the non-inclusion game.

Winning region of refuter

Theorem

The set of rejecting positions

$$W^{\not\subseteq} \;=\; \{\alpha \in \vartheta \mid \varphi(F_\alpha) = 1\}$$

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Proof

Position $w \in \mathcal{L}(A)$ has formula $F_w = [w]$ with $\varphi([w]) = 0$

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The set of rejecting positions

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Proof

Position $w \in \mathcal{L}(A)$ has formula $F_w = [w]$ with $\varphi([w]) = 0$

$\Rightarrow \mathcal{L}(A) \cap W^{\not\subseteq} = \emptyset$

Winning region of refuter

Theorem

The set of rejecting positions

$$W^{\not\subseteq} \;=\; \{\alpha \in \vartheta \mid \varphi(F_\alpha) = 1\}$$

is the winning region of refuter $\bigcirc$ for the non-inclusion game.

Proof

Position $w \in \mathcal{L}(A)$ has formula $F_w = [w]$ with $\varphi([w]) = 0$

$\Rightarrow \mathcal{L}(A) \cap W^{\not\subseteq} = \emptyset$

Show: If the current position is rejecting and it is the turn of

Winning region of refuter

Theorem

The set of rejecting positions

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Position $w \in \mathcal{L}(A)$ has formula $F_w = [w]$ with $\varphi([w]) = 0$

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Show: If the current position is rejecting and it is the turn of

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Show: If the current position is rejecting and it is the turn of

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$\Rightarrow \mathcal{L}(A) \cap W^{\not\subseteq} = \emptyset$

Show: If the current position is rejecting and it is the turn of

(1) Refuter: There is a move to a rejecting position,

(2) Prover: All moves go to rejecting positions.

Not sufficient to win reachability game, need to minimize distance to $\overline{\mathcal{L}(A)}$ in every step.

Winning region of refuter

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The set of rejecting positions

$$W^{\not\subseteq} \;=\; \{\alpha \in \vartheta \mid \varphi(F_\alpha) = 1\}$$

is the winning region of refuter ○ for the non-inclusion game.

In the example, starting from Y:

$[b]$ does not contain (q_0, q_0)

Winning region of refuter

Theorem

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$$W^{\not\subseteq} \;=\; \{\alpha \in \vartheta \mid \varphi(F_\alpha) = 1\}$$

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In the example, starting from Y:

$[b]$ does not contain (q_0, q_0)

↳ $\varphi(F_Y) = \varphi([b]) = 1$

Winning region of refuter

Theorem

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↳ Y is rejecting

Winning region of refuter

Theorem

The set of rejecting positions

$$W^{\not\subseteq} \;=\; \{\alpha \in \vartheta \mid \varphi(F_\alpha) = 1\}$$

is the winning region of refuter $\bigcirc$ for the non-inclusion game.

In the example, starting from Y:

$[b]$ does not contain (q_0, q_0)

↳ $\varphi(F_Y) = \varphi([b]) = 1$

↳ Y is rejecting

Indeed, refuter wins non-inclusion from Y

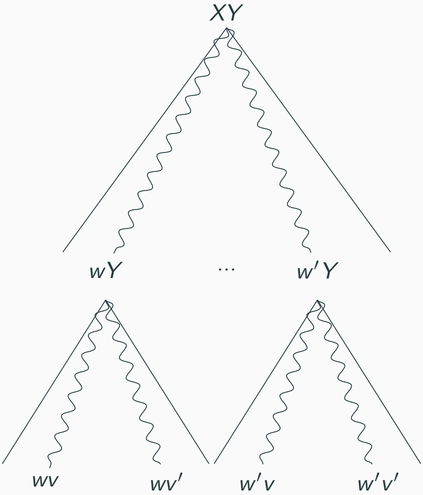
Composition

Composition

How to define the composition operator ; that replaces concatenation . in the system of equations?

Composition

Plays from XY decompose:



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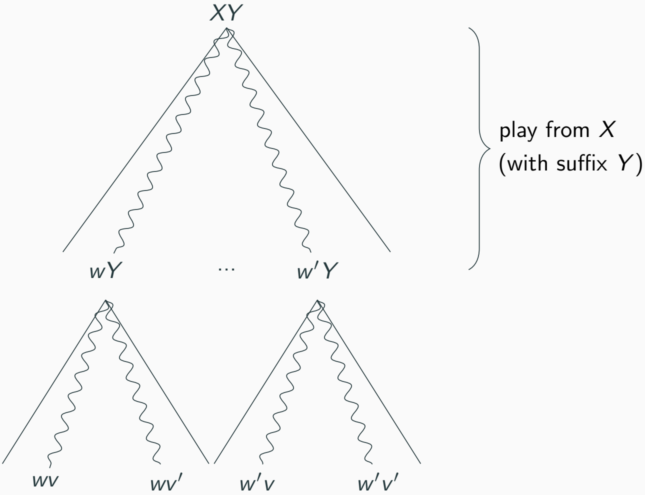

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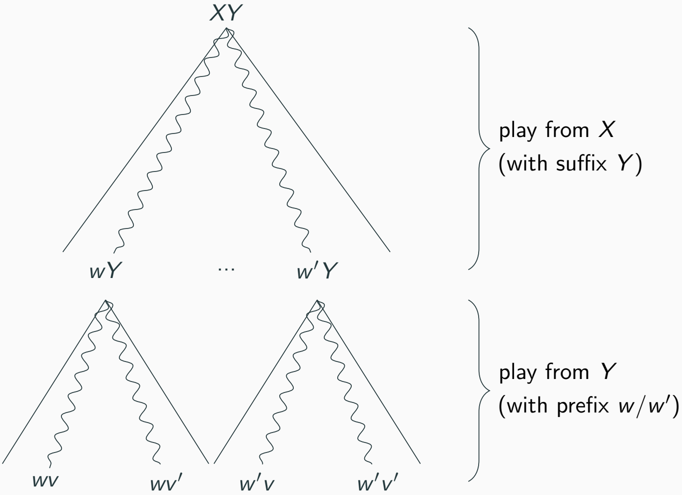

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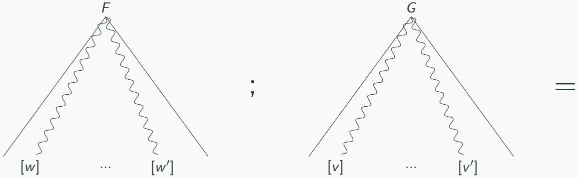

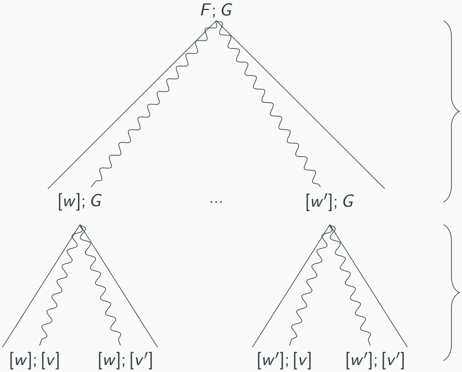

$(F * F'); G = F; G * F'; G$

$[w]; (G * G') = [w]; G * [w]; G'$

Complexity & Performance

Algorithm

Given: Game G, A and initial position α

Algorithm for solving non-inclusion:

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(1) Set $F_X = \mathit{false}$ for all $X \in N$

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Compose solutions F_X for non-terminals to obtain the solutions for all sentential forms $\alpha = \alpha_1 \dots \alpha_k \in \vartheta$: $F_\alpha \; = \; F_{\alpha_1}; \dots ; F_{\alpha_k}$

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Solve system once and decide game for any position α

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Theorem

1. *Deciding non-inclusion games is* 2EXPTIME*-complete.*

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2. *The algorithm solves non-inclusion games in*

$$\mathcal{O}\left(|G|^2 \cdot 2^{2^{|Q|^{c_1}}} + |\alpha| \cdot 2^{2^{|Q|^{c_2}}}\right)$$

where $c_1, c_2 \in \mathbb{N}$ *are constants.*

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3. *Hardness by reduction from acceptance in alternating Turing machines with exponential space.*

Related Work

Cachat [C02]:

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Consider pushdown system with ownership partitioning of control states

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Can one player enforce a configuration such that the stack content is accepted by an alternating finite automaton (AFA)?

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EXPTIME

↳ Our game **can be reduced** to Cachat

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EXPTIME

↳ Similar technique **can be applied** to our problem

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2EXPTIME for left-to-right strategies

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- 2EXPTIME for left-to-right strategies
- Similar to our game

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Undecidable

- 2EXPTIME for left-to-right strategies
- Similar to our game
- Hardness proof **carries over**

Performance

Comparison of 2EXPTIME *algorithms:*

Input		Computation	
Our algorithm			
System of equations	P	Fixed-point iteration	2EXP
Reduction to Cachat [C02]			
Determinized automaton	EXP	Saturation	EXP
Idea of Walukiewicz [W01]			
Finite reachability game	2EXP	Saturation	P

guaranteed blow-up (Input) — may be lucky (Computation)

Performance

We have implemented and compared:

- Our algorithm with naive Kleene iteration
- Our algorithm with worklist-based Kleene iteration
- Reduction to Cachat's pushdown games

Problems with Cachat's algorithm:

- Automaton A needs to be determinized
  ↳ Guaranteed blow-up
- Algorithmic tricks for Cachat (worklist, ...) not suitable for the instances generated by the reduction

Performance

	naive Kleene		worklist Kleene		Cachat	
$\lvert Q\rvert/\lvert N\rvert/\lvert T\rvert$	avg. time	% timeout	avg. time	% timeout	avg. time	% timeout
5/ 5/ 5	65.2	2	0.8	0	94.7	0
5/ 5/10	5.4	4	7.4	0	701.7	0
5/10/ 5	13.9	0	0.3	0	375.7	0
5/ 5/15	6.0	0	1.1	0	1618.6	0
5/10/10	32.0	2	122.1	0	2214.4	0
5/15/ 5	44.5	0	0.2	0	620.7	0
5/ 5/20	3.4	0	1.4	0	3434.6	4
5/10/15	217.7	0	7.4	0	5263.0	16
10/ 5/ 5	8.8	2	0.6	0	2737.8	2
10/ 5/10	9.0	6	69.8	0	6484.9	66
15/ 5/ 5	30.7	0	0.2	0	5442.4	52
10/10/ 5	9.7	0	0.2	0	7702.1	92
10/15/15	252.3	0	1.9	0	n/a	100
10/15/20	12.9	0	1.8	0	n/a	100

Experiments executed on i7-6700K, 4GHz, times in milliseconds, timeout 10 seconds

Future Work

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Liveness synthesis (infinite words)

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Synthesis for systems with branching behavior (trees)

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Games on higher-order systems

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Applications in hardware synthesis

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Applications in hardware synthesis

Solver technology for systems of equations (Newton iteration)

Questions?