16. Fixed Points

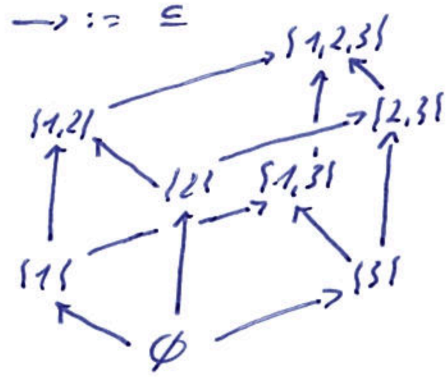
<u>Goal</u>: · Introduce basic notions from lattice theory
· Present the main fixed point theorems.

16.1 Complete Lattices

<u>Goal</u>: · Define partial orders, join and meet, lattices, complete lattices, bottom and top.

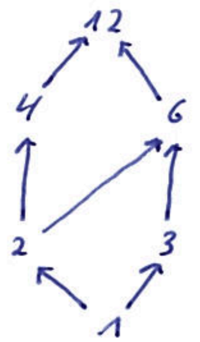
<u>Observation</u>: · $(\mathbb{N}, \leq)$ is totally ordered, every two elements are comparable.
· Some domains are only <u>partially</u> ordered:

Subsets of $\{1,2,3\}$, $\rightarrow := \subseteq$



$\{1,2\}$ and $\{2,3\}$ are incomparable

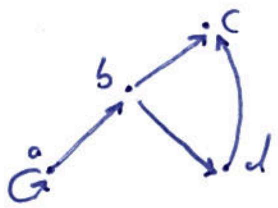
Divisors of 12, $\rightarrow := |$



2 and 3 are incomparable.

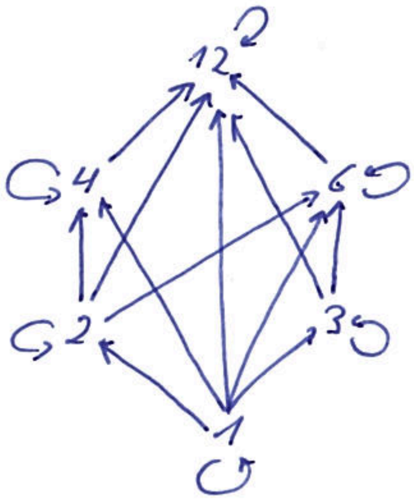
<u>Definition</u>:
A <u>partial</u> <u>order</u> $(D, \leq)$ consists of a set $D \neq \emptyset$ and a reflexive, transitive, and anti-symmetric relation $\leq \subseteq D \times D$.
Recall that anti-symmetry requires $\forall x, y \in D: x \leq y \wedge y \leq x \Rightarrow x = y$.

· Binary relations can be understood as <u>directed</u> <u>graphs</u>: $\{(a,a), (a,b), (b,c), (b,d), (d,c)\}$ =



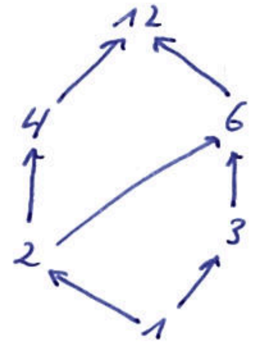
Partial orders yield particular graphs:
↳ Reflexivity = Loops at every node
↳ Anti-Symmetry = No cycles (except for loops)
↳ Transitivity = Transitivity of the edges.

Example (Divisors of 12):



Hasse diagrams of partial orders do not draw

- loops and
- induced edges:



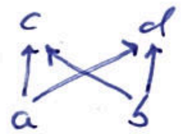
Definition (Join and meet):

Let $(D, \leq)$ be a partial order.

- An element $\sigma \in D$ is an upper bound of a set $X \subseteq D$, if $x \leq \sigma$ for all $x \in X$.
- An element $\sigma \in D$ is a least upper bound, also called join of $X \subseteq D$, if
  - σ is an upper bound of X and
  - $\sigma \leq \sigma'$ for every upper bound σ' of X.

  We write $\bigsqcup X$ for the join of X.
- Similarly, $u = \bigsqcap X$ is the greatest lower bound, also called meet of X.

Example:



a and b:

- have c and d as upper bounds
- but do not have a least upper bound.

Definition (Complete lattice):

- A lattice is a partial order $(D, \leq)$ where join $a \sqcup b$ (*) and meet $a \sqcap b$ exist for all $a, b \in D$.
- A lattice is complete if every subset $X \subseteq D$ has join and meet.

(*) The infix notation stands for $\bigsqcup \{a, b\}$.

Example: (1) a $\cdot$ $\cdot$ b = no lattice (2) $0 \to 1 \to 2 \cdots$ = no complete lattice.

Lemma:

(1) Every complete lattice $(D, \leq)$ has a unique

↳ least element $\bot := \bigsqcup \emptyset = \bigsqcap D$

↳ greatest element $\top := \bigsqcap \emptyset = \bigsqcup D$

(2) Every finite lattice $(D, \leq)$ (with D finite) is complete.

16.2 Monotone Functions and Knaster and Tarski's Theorem:

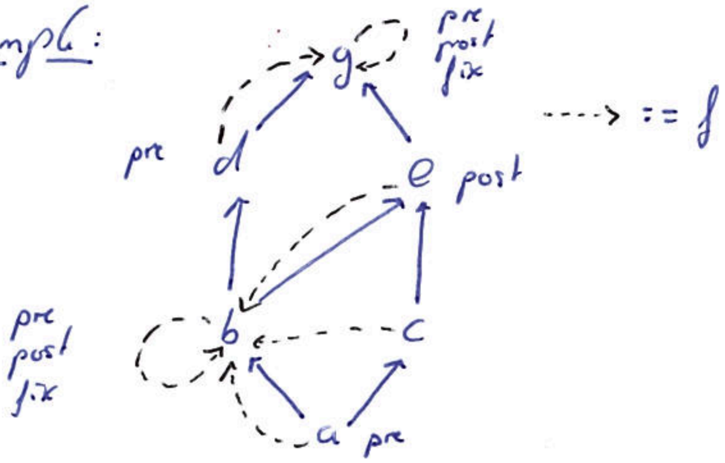
Goal: Show the existence of fixed points for monotone functions on complete lattices.

Definition (Monotone functions and fixed points):

Let $(D, \leq)$ be a partial order.

- A function $f: D \to D$ is monotone, if $x \leq y$ implies $f(x) \leq f(y)$ for all $x, y \in D$.
- A fixed point of $f: D \to D$ is an element $x \in D$ with $f(x) = x$.
  We use $Fix(f) := \{x \in D \mid f(x) = x\}$ for the set of all fixed points of f.
- A pre-fixed point is an element $x \in D$ with $x \leq f(x)$.
  We use $Prefix(f)$ for the set of all pre-fixed points of f.
- A post-fixed point of f is an element $x \in D$ with $f(x) \leq x$.
  We use $Postfix(f)$ for the set of all post-fixed points of f.

Example:



Theorem (Knaster and Tarski '55):

Let $(D, \leq)$ be a complete lattice and
let $f : D \to D$ be monotone.

(1) Then $\sqcap Postfix(f)$ is a fixed point of f.
Moreover, it is the least fixed point,
and commonly denoted by $lfp(f)$.

(2) Similarly, $\sqcup Prefix(f)$ is the greatest fixed point of f, $gfp(f)$.

Proof:
We show (1), the reasoning for (2) is by duality (turn around the lattice.

- It is sufficient to show that $\sqcap Postfix(f)$ is a fixed point.
  Since every fixed point is a post-fixed point,
  $\sqcap Postfix(f)$ has to be smaller than every fixed point of f.

- We use $l := \sqcap Postfix(f)$ to denote the meet.
  We first show

  $$f(l) \leq l.$$

  Since $l \leq l'$ for all $l' \in Postfix(f)$,
  and since f is monotone, we obtain

  $$f(l) \leq f(l') \underset{(Postfix)}{\leq} l' \quad \text{for all } l' \in Postfix(f).$$

This means $f(l)$ is a lower bound for Postfix(f).

Since l is the greatest lower bound for Postfix(f),

$$f(l) \le l \qquad (*)$$

follows.

- We now show

$$l \le f(l).$$

By $(*)$ and monotonicity, we have

$$f(f(l)) \le f(l).$$

But this means $f(l) \in$ Postfix(f).

Hence,

$$l \le f(l). \qquad (**)$$

- Anti-symmetry allows us to conclude $l = f(l)$ from $(*)$ and $(**)$. □

Note:

The above actually is a weak version of Knaster and Tarski's theorem.
The full version states that $(\mathrm{Fix}(f), \le)$, i.e., the set of all fixed points,
again forms a complete lattice.

16.3 Chains, Continuous Functions, and Kleene's Theorem

Goal: · Knaster and Tarski's result only tells us
that fixed points exist (for monotone functions on complete lattices).

· Kleene's theorem will allow us to compute them by iteration.

Definition:

Let $(D, \le)$ be a partial order.

· A chain is a totally ordered subset $K \subseteq D$.

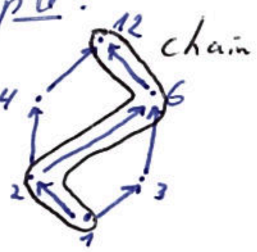
We also give chains as sequences (hi linear consisting of the members.

- A sequence $(k_i)_{i\in\mathbb{N}}$ is an <u>ascending</u> <u>chain</u>,
  if $k_i \le k_{i+1}$ for all $i\in\mathbb{N}$.
- A sequence $(k_i)_{i\in\mathbb{N}}$ is a <u>descending</u> <u>chain</u>,
  if $k_i \ge k_{i+1}$ for all $i\in\mathbb{N}$.
- An ascending/descending chain $(k_i)_{i\in\mathbb{N}}$ becomes <u>stationary</u>,
  if $\exists n\in\mathbb{N}\ \forall i\ge n: k_i = k_n$.

  One also says the chain <u>stabilizes</u>.
- $(D,\le)$ has <u>finite</u> <u>height</u>, if every chain $K\subseteq D$ has finitely many elements.
- $(D,\le)$ has <u>bounded</u> <u>height</u>, if there is an $n\in\mathbb{N}$ so that every chain $K\subseteq D$ has at most n elements.

<u>Example</u>:

(1)



(2) In $(\mathbb{N},\le)$, every descending chain becomes stationary.

<u>Definition (Chain conditions)</u>:

A partial order $(D,\le)$

↳ satisfies the <u>ascending</u> <u>chain</u> <u>condition</u> <u>(ACC)</u>
(one also says $(D,\le)$ is an <u>Artinian</u> <u>lattice</u> (Emil Artin, 1898-1962))
if every ascending chain $k_0\le k_1\le\dots$ becomes stationary;

↳ satisfies the <u>descending</u> <u>chain</u> <u>condition</u> <u>(DCC)</u>
($(D,\le)$ is <u>Noetherian</u> (Emmy Noether, 1882 - 1935))
if every descending chain $k_0\ge k_1\ge\dots$ becomes stationary.

<u>Note</u>: (ACC) and (DCC) are independent of the lattice conditions.

Lemma:
A partial order has finite height iff (ACC) and (DCC) hold.

Definition (Continuity):
Let $(D, \leq)$ be a complete lattice.
A function $f: D \to D$ is
(i) $\sqcup$-continuous, if for every chain $K \subseteq D$:
(join-continuous, upward-continuous)
$$f(\bigsqcup K) = \bigsqcup f(K) := \bigsqcup \{ f(k) \mid k \in K \}.$$

(ii) $\sqcap$-continuous, if for every chain $K \subseteq D$:
(meet-continuous, downward-continuous)
$$f(\bigsqcap K) = \bigsqcap f(K) := \bigsqcap \{ f(k) \mid k \in K \}.$$

Our goal is to show that monotonicity implies $\overset{\sqcup}{\sqcap}$-continuity, provided the corresponding chain condition holds.
The proof relies on the existence of greatest and least elements.

Lemma:
Let $(D, \leq)$ be a partial order.
(i) If (ACC) holds, every non-empty chain $K \subseteq D$ has a greatest element $g \in K$ (with $k \leq g$ for all $k \in K$).
(ii) If (DCC) holds, every non-empty chain $K \subseteq D$ contains a least element $l \in K$ (with $l \leq k$ for all $k \in K$).

Proof:
We show (i), (ii) is by duality.
Towards a contradiction, assume
$$\neg (\exists g \in K \; \forall k \in K : k \leq g)$$
which means
$$\forall g \in K \; \exists k \in K : \; g \text{ and } k \text{ are incomparable or } g < k. \qquad (*)$$

Case (†) cannot hold:

Since both g and h stem from K, and since K is a chain,

they have to be comparable.

We thus have

$$\forall g \in K \; \exists h \in K: g < h \quad (g \leq h \text{ and } g \neq h). \qquad (**)$$

This contradicts (ACC), as can be seen from the following ascending chain that does not become stationary:

$h_0 :=$ an arbitrary element from K

$h_{i+1} :=$ the $h \in K$ with $h > h_i$,

which exists by $(**)$. □

<u>Theorem (Monotonicity implies continuity):</u>

Let $(D, \leq)$ be a complete lattice and let $f: D \to D$ be monotone.

(i) If $(D, \leq)$ satisfies (ACC), then f is $\sqcup$-continuous.

(ii) If $(D, \leq)$ satisfies (DCC), then f is $\sqcap$-continuous.

<u>Proof:</u>

We again only show (i). Consider an ascending chain $K \subseteq D$.

- To see that $\bigsqcup f(K) \leq f(\bigsqcup K)$,

  note that $h \leq \bigsqcup K$ for all $h \in K$.

  Hence, $f(h) \leq f(\bigsqcup K)$ by monotonicity.

  Since this holds for all $h \in K$,

  $f(\bigsqcup K)$ is an upper bound for $f(K) = \{ f(h) \mid h \in K \}$.

  Hence, the least upper bound is smaller, $\bigsqcup f(K) \leq f(\bigsqcup K)$.

- To see that $f(\bigsqcup K) \leq \bigsqcup f(K)$,

  note that $\bigsqcup K = g$ with g the greatest element from the above lemma.

  Now $f(\bigsqcup K) = f(g) \leq \bigsqcup f(K)$.

  The inequality holds as $g \in K$.

- We conclude by anti-symmetry. □

For Kleene's fixed point theorem, we consider chains obtained by <u>iterating</u> the function of interest on $\bot$.
This gives us a way to actually <u>compute</u> the least fixed point.

<u>Lemma:</u>
Let $(D, \leq)$ be a complete lattice and let $f: D \to D$ be monotone.
The sequence
$$(f^i(\bot))_{i \in \mathbb{N}} \quad \text{with } f^0(\bot) := \bot \text{ and } f^{i+1}(\bot) := f(f^i(\bot))$$
is an ascending chain.

<u>Proof:</u>
We proceed by induction and show that $f^i(\bot) \leq f^{i+1}(\bot)$ for all $i \in \mathbb{N}$.

<u>Base case</u>: $f^0(\bot) = \bot \leq f(\bot)$, since bottom is the least element of D, $\bot = \sqcap D$.

<u>Induction step</u>: Assume $f^i(\bot) \leq f^{i+1}(\bot)$.
Then
$$f^{i+1}(\bot) = f(f^i(\bot)) \underset{\substack{(IH+ \\ \text{Monotonicity})}}{\leq} f(f^{i+1}(\bot)) = f^{i+2}(\bot).$$
$\square$

<u>Theorem (Kleene):</u>
Let $(D, \leq)$ be a complete lattice and $f: D \to D$ monotone.
(i) If f is $\sqcup$-continuous, $lfp(f) = \sqcup\{f^i(\bot) \mid i \in \mathbb{N}\}$.
(ii) If f is $\sqcap$-continuous, $gfp(f) = \sqcap\{f^i(\top) \mid i \in \mathbb{N}\}$.

<u>Proof:</u>
We again show (i).
- The first step is to prove that $\sqcup\{f^i(\bot) \mid i \in \mathbb{N}\}$ is a fixed point:
$$f(\sqcup\{f^i(\bot) \mid i \in \mathbb{N}\}) \overset{(\sqcup\text{-continuity})}{=} \sqcup\{f^{i+1}(\bot) \mid i \in \mathbb{N}\} \overset{(\bot = \sqcap D)}{=} \sqcup\{f^i(\bot) \mid i \in \mathbb{N}\}.$$

- We now show that $\bigsqcup\{f^i(\bot) \mid i \in \mathbb{N}\}$ is the least fixed point.
  To this end, we consider $d \in D$ with $f(d) = d$
  and show that $\bigsqcup\{f^i(\bot) \mid i \in \mathbb{N}\}$ is smaller.
- By an induction on $i \in \mathbb{N}$, we prove $f^i(\bot) \leq d$ for all $i \in \mathbb{N}$.

  Base case: $f^0(\bot) = \bot \leq d$, since $\bot = \sqcap D$.

  Induction step: Assume $f^i(\bot) \leq d$.
  We then get
  $$f^{i+1}(\bot) = f(f^i(\bot)) \underset{\substack{(IH + \\ \text{Monotonicity})}}{\leq} f(d) = d.$$
- Since $f^i(\bot) \leq d$ for all $i \in \mathbb{N}$, d is an upper bound for $\{f^i(\bot) \mid i \in \mathbb{N}\}$,
  and thus $\bigsqcup\{f^i(\bot) \mid i \in \mathbb{N}\} \leq d$. $\square$

In combination with the above theorem on monotonicity implying continuity under appropriate chain conditions, we obtain the following.

Corollary:
Let $(D, \leq)$ be a complete lattice with (ACC) and (DCC).
Let $f : D \to D$ be monotone.
Then
- $lfp(f) = \bigsqcup\{f^i(\bot) \mid i \in \mathbb{N}\}$
  $= f^n(\bot)$ with $f^n(\bot) = f^{n+1}(\bot)$.
- $gfp(f) = \sqcap\{f^i(\top) \mid i \in \mathbb{N}\}$
  $= f^n(\top)$ with $f^n(\top) = f^{n+1}(\top)$.

Recall that (ACC) and (DCC) hold iff the lattice has finite height.