4.3.2 Complementation algorithm

Let $A = (Q, q_0, \rightarrow, Q_F)$ an NBA.

Our goal is to construct an NBA $\bar{A}$ with $L(\bar{A}) = \overline{L(A)}$

Key idea:

Define an equivalence $\sim_A \subseteq \Sigma^* \times \Sigma^*$ on words depending on "how they move in A"

↳ coarse enough to have finitely many classes

↳ fine enough to capture what is/is not in $L(A)$ by classes.

Recall:

$q \xrightarrow{u} q'$ means there are states $q_1, \dots, q_n$ so that

$$q \xrightarrow{a_0} q_1 \xrightarrow{a_1} \dots \xrightarrow{a_n} q \quad \text{and} \quad u = a_0 \dots a_n.$$

Define

$q \xrightarrow{u}_{fin} q'$ if $q \xrightarrow{u} q'$ so that at least one intermediary state is final.

Observation:

- $q \xrightarrow{u}_{fin} q'$ implies $q \xrightarrow{u} q'$ and
- $q \xrightarrow{u} q_f$ and $q_f \xrightarrow{v} q'$ with $q_f \in Q_F$ then $q \xrightarrow{u.v}_{fin} q'$.

Definition (Transition equivalence):

Transition equivalence $\sim_A \subseteq \Sigma^* \times \Sigma^*$ is defined by

$u \sim_A v$ if for all $q, q' \in Q$ we have

$$q \xrightarrow{u} q' \text{ iff } q \xrightarrow{v} q' \text{ and}$$
$$q \xrightarrow{u}_{fin} q' \text{ iff } q \xrightarrow{v}_{fin} q'.$$

Intuitively:

Equivalence $u \sim_A v$ means u and v yield the same state changes in A

(even when considering intermediary final states).

As there are only finitely many states in A,
equivalence $\sim_A$ has finite index.

<u>Lemma:</u>

For every NBA $A = (Q, q_0, \rightarrow, Q_F)$, equivalence $\sim_A \subseteq \Sigma^* \times \Sigma^*$
has finitely many classes.

<u>Proof:</u>

First condition : $|Q|^2$ pairs of states

Second condition : $|Q|^2$ pairs of state

Choices of whether $q \xrightarrow{u} q'$ and $q \xrightarrow{u}_{fin} q'$:

$2^{2|Q|^2}$ many equivalence classes. $\square$

<u>Lemma:</u> Consider an arbitrary NBA A.
Every equivalence class $[u]_{\sim_A} = \{v \in \Sigma^* \mid u \sim_A v\}$
is a regular language.

The technique used in the proof is important.

<u>Proof:</u>

Let $A = (Q, q_0, \rightarrow, Q_F)$.

For $q, q' \in Q$ define two languages

$$L_{q,q'} := \{u \in \Sigma^* \mid q \xrightarrow{u} q'\}$$

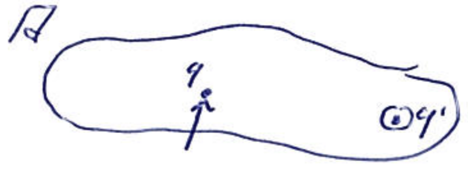
$$L^{fin}_{q,q'} := \{u \in \Sigma^* \mid q \xrightarrow{u}_{fin} q'\}.$$

Both languages are regular:

$L_{q,q'} = L(A_{q,q'})$ with $A_{q,q'} = (Q, q, \rightarrow, \{q'\})$

// q as initial state, q' as final state

// $A_{q,q'}$ a finite automaton



$L^{fin}_{q,q'} = \mathcal{L}(\mathcal{A}^{fin}_{q,q'})$ with $\mathcal{A}^{fin}_{q,q'} = (Q \times \{0,1\}, (q,h), \to', \{(q',1)\})$

where $h = \begin{cases} 0 & \text{if } q \notin Q_F \\ 1 & \text{otherwise} \end{cases}$ and $(\hat{q}, i) \xrightarrow{a}' (\tilde{q}, j)$ if $\hat{q} \xrightarrow{a} \tilde{q}$

// set flag to 1
// when initial
state is final

and $j = \begin{cases} 0 & \text{if } i = 0 \text{ and } q' \notin Q_F \\ 1 & \text{otherwise} \end{cases}$

// q as initial state, q' as final state
// $\mathcal{A}$ a finite automaton
// set flag to 1 when final state found
// only accept with flag

We have

$$\boxed{[u]_{\sim_{\mathcal{A}}} = \bigcap_{q,q' \in Q} \widetilde{L_{q,q'}} \cap \widetilde{L^{fin}_{q,q'}}}$$

where

$$\widetilde{L_{q,q'}} := \begin{cases} L_{q,q'} & \text{if } q \xrightarrow{u} q' \\ \overline{L_{q,q'}} & \text{otherwise} \end{cases}$$

$$\widetilde{L^{fin}_{q,q'}} := \begin{cases} L^{fin}_{q,q'} & \text{if } q \xrightarrow{u}_{fin} q' \\ \overline{L^{fin}_{q,q'}} & \text{otherwise.} \end{cases}$$

- As the set of states in $\mathcal{A}$ is finite, so is the intersection.
- We argued that $L_{q,q'}$ and $L^{fin}_{q,q'}$ are regular languages.
  - ↳ Regular languages are closed under complementation
  - ↳ and closed under finite intersections.

So $[u]_{\sim_{\mathcal{A}}}$ is a regular language.

$\square$

Although equivalence $\sim_A$ has finitely many classes it is fine enough so that its classes
↳ either fully belong to $\mathcal{L}(A)$ or
↳ do not intersect $\mathcal{L}(A)$.

<u>Lemma</u>:

Consider an NBA A, two classes $[u]_{\sim_A}$ and $[v]_{\sim_A}$ of $\sim_A$, and $w \in [u]_{\sim_A}.([v]_{\sim_A})^\omega$ an ω-word.

If $w \in \mathcal{L}(A)$ then $[u]_{\sim_A}.([v]_{\sim_A})^\omega \subseteq \mathcal{L}(A)$.

<u>Proof</u>:

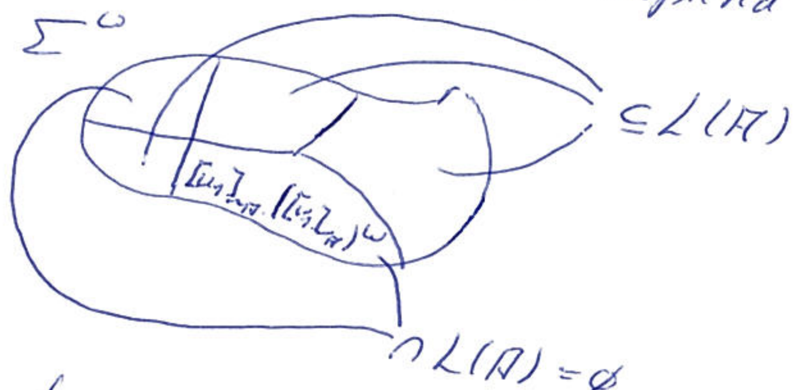
Homework.

<u>Corollary</u>:

Let A an NBA, $[u]_{\sim_A}$ and $[v]_{\sim_A}$ two classes of $\sim_A$, and $w \in [u]_{\sim_A}.([v]_{\sim_A})^\omega$ an ω-word.

If $w \in \overline{\mathcal{L}(A)}$ then $[u]_{\sim_A}.([v]_{\sim_A})^\omega \subseteq \overline{\mathcal{L}(A)}$.

We now show that every word in Σ^ω falls into such a composition of equivalences classes $[u]_{\sim_A}.([v]_{\sim_A})^\omega$.
As a consequence, Σ^ω can be interpreted as



The proof is an application of Ramsey's theorem.

<u>Lemma</u>:

Consider an NBA A. For every word $w \in \Sigma^\omega$ there are classes $[u]_{\sim_A}$ and $[v]_{\sim_A}$ so that $w \in [u]_{\sim_A}.([v]_{\sim_A})^\omega$.

Proof:

Let $w = a_0 a_1 a_2 \ldots \in \Sigma^\omega$.

Consider the following coloring of (V, E) with $V = \mathbb{N}$.

Let

$$f(\{i,j\}) := [a_i \ldots a_{j-1}]_{\sim_A} \quad \text{with } i < j.$$

Since $\sim_A$ has only finitely many classes, Ramsey's theorem applies and gives

- an equivalence class $[v]_{\sim_A}$ and
- an infinite subset $S \subseteq \mathbb{N}$

so that

$$f(\{i,j\}) = [v]_{\sim_A} \quad \text{for all } i < j \text{ in } S.$$

This means

$[a_i \ldots a_{j-1}]_{\sim_A} = [v]_{\sim_A}$, so $a_i \ldots a_{j-1} \sim_A v$,

which means $a_i \ldots a_{j-1} \in [v]_{\sim_A}$.

Let $i_0 \in S$ minimal.

Then

$$w \in [a_0 \ldots a_{i_0 - 1}]_{\sim_A} . ([v]_{\sim_A})^\omega.$$

Note that every word $a_0 \ldots a_{i_0-1}$ belongs to its own equivalence class, $a_0 \ldots a_{i_0-1} \in [a_0 \ldots a_{i_0-1}]_{\sim_A}$.

$\square$

Theorem (Büchi '62)

Let A an NBA. Then $\overline{L(A)}$ is effectively ω-regular.

Proof:

$$\overline{L(A)} = \bigcup_{[u]_{\sim_A} . ([v]_{\sim_A})^\omega \cap L(A) = \emptyset} [u]_{\sim_A} . ([v]_{\sim_A})^\omega$$

Note that there are finitely many classes.

Thus this language is ω-regular.

Effectiveness:

- Determine all classes $[u]_{\sim_A}$ by automata constructions:
  - ↳ Pic the state changes $q \xrightarrow{u} q'$ that should / should not hold for the class.
  - ↳ Construct the corresponding automata $A_{q,q'}$ and $A^{fin}_{q,q'}$ (or their complements)
  - ↳ Intersect the languages as stated in Lemma above.
- ω-iteration of regular languages and concatenation of regular with ω-regular languages can be (effectively) performed on the corresponding automata. So $[u]_{\sim_A}.([v]_{\sim_A})^\omega$ can be represented (effectively) by an NBA $A_{[u]_{\sim_A}.([v]_{\sim_A})^\omega}$.
- Intersection of $L(A_{[u]_{\sim_A}.([v]_{\sim_A})^\omega})$ with $L(A)$ can be computed by parallel composition.
- Emptiness of $L(A_{[u]_{\sim_A}.([v]_{\sim_A})^\omega} \| A)$ decidable.
- Finite union of ω-regular languages is ω-regular. $\square$

By theorem in homework, we can as well represent $\bigcup_{\dots} [u]_{\sim_A}.([v]_{\sim_A})^\omega$ by an NBA.

Corollary:

Given an NBA A, we can effectively construct an NBA $\bar{A}$ with $L(\bar{A}) = \overline{L(A)}$.

Later on, we give a direct construction.