René Maseli Prof. Dr. Roland Meyer	**Theoretical Computer Science 1** **Exercise 1**	TU Braunschweig Winter semester 2022/23

Release: 01.11.2022 Due: 11.11.2022, 09:45

Hand in your solutions to the Vips directory of the StudIP course until Friday, 11.11.2022 09:45 pm. You should provide your solutions either directly as .pdf file or as a readable scan/photo of your handwritten notes. Submit your results as a group of four.

Exercise 1: [12 points]

Let $(\mathbb{N}, \preceq)$ be a lattice, where $\preceq$ is a binary relation over $\mathbb{N}$ defined as follows: For $x, y \in \mathbb{N}$ the pair $x \preceq y$ holds if and only if $x = 0$ or $y = 1$ or $x = y \in \mathbb{N} \setminus \{0, 1\}$.

- [1 point] Draw a Hasse-diagram of $(\mathbb{N}, \preceq)$ for the numbers up to 9.
- [1 point] State $\top$ und $\bot$ of this lattice.
- [7 points] State the values of the following joins and meets:
  - $\bot \sqcup \top$
  - $\bot \sqcap \top$
  - $\top \sqcup 5$
  - $6 \sqcap 7$
  - $\bot \sqcup 4$
  - $\bigsqcup\{n \in \mathbb{N} \mid n \text{ is even}\}$
- [2 points] Is the height of this lattice finite? Is it bounded?
- [2 points] Give a Hasse-diagram for a lattice which has finite but non-bounded height.

Exercise 2: [9 points]

Let $M_1 \subseteq \mathbb{N}$ and $M_2 \subseteq \mathbb{N}$ be two finite sets and $M = M_1 \times M_2$ the set of all pairs (a, b) with $a \in M_1$ and $b \in M_2$. Let $\preceq$ be a relation on M, defined as follows:

$$(a_1, b_1) \preceq (a_2, b_2) \quad \text{if and only if} \quad a_1 \geqslant a_2 \text{ and } b_1 \geqslant b_2$$

where $\leqslant$ is the common "less or equals" relation on natural numbers.

- [3 points] Show that $\preceq$ is reflexive, transitive and antisymmetrical.

  By definition, $(M, \preceq)$ is then a partial order.

- [4 points] Show that the join $\bigsqcup M'$ and the meet $\bigsqcap M'$ exist for each subset $M' \subseteq M$.

  By definition, $(M, \preceq)$ is then a complete lattice.

- [1 point] State $\top, \bot$ for this lattice.

- [1 point] Does $(M, \preceq)$ stay complete, if $M_1 \subseteq \mathbb{N}$ is infinite?

Exercise 3: Product Lattice [8 points]

a) [4 points] Let $(D_1, \preceq_1)$ and $(D_2, \preceq_2)$ be complete lattices. The **product lattice** is defined as $(D_1 \times D_2, \preceq)$, where $\preceq$ is the **product ordering** on tuples with $(d_1, d_2) \preceq (d_1', d_2')$ if and only if $d_1 \preceq_1 d_1'$ and $d_2 \preceq_2 d_2'$.

   Show that the product lattice is indeed a complete lattice.

b) [4 points] Prove the following; The product lattice $(D_1 \times D_2, \preceq)$ satisfies ACC if and only if $(D_1, \preceq_1)$ and $(D_2, \preceq_2)$ both satisfy ACC.

Exercise 4: Distributivity [6 points]

Let $(D, \leqslant)$ be a lattice and $x, y \in D$ be two arbitrary elements.

a) [3 points] Show that if $f : D \to D$ is monotone, then $f(x \sqcup y) \geqslant f(x) \sqcup f(y)$ holds.

b) [3 points] $f : D \to D$ is called **distributive**, if $f(x \sqcup y) = f(x) \sqcup f(y)$ for all $x, y \in D$.

   Show that if f is distributive then f is also monotone.