Theoretical Computer Science 1
Exercise Sheet 2

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Hand in your solutions to the Vips directory of the StudIP course until Thursday, November 23th 2023 23:59. You should provide your solutions either directly as .pdf file or as a readable scan/photo of your handwritten notes. Submit your results as a group of four and state **all** members of your group with **student id, name and course**.

Homework Exercise 1: Join-Meet-Continuity [5 points]
Prove or disprove, if the following statements hold.

a) [2 points] Let $n_1, n_2, n_3 \in \mathbb{N} \setminus \{0\}$ be positive integers and $D := \{\, k \in \mathbb{N} \mid k \mid n_1 \cdot n_2 \cdot n_3 \,\}$ the set of divisors of the product $n_1 \cdot n_2 \cdot n_3$. Consider the lattice $\langle D, \mid \rangle$. Show that the function $f_a : D \to D$ is $\sqcap$-continuous.

$$f_a(x) = \gcd(n_1, \operatorname{lcm}(n_2, n_3 \cdot x))$$

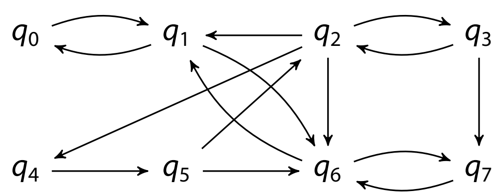
b) [1 point] Let M be a finite set, $R \subseteq M \times M$ a binary relation over M. Consider the complete lattice $\langle \mathcal{P}(M), \subseteq \rangle$. Show that the function $f_b : \mathcal{P}(M) \to \mathcal{P}(M)$ is **not** $\sqcup$-continuous.

$$f_b(X) = \{\, y \in M \mid \forall x \in M : \langle x, y \rangle \in R \Rightarrow x \in X \,\}$$

c) [2 points] Let $\langle B, E, \to \rangle$ be a control flow graph. Consider the lattice $\langle \mathcal{P}(\mathrm{Var})^B, \subseteq^B \rangle$, where all elements $F, G : B \to \mathcal{P}(\mathrm{Var})$ are ordered by $F \subseteq^B G$ iff $\forall b \in B : F(b) \subseteq G(b)$. A block is said to *use* a variable, if the variable appears inside any expression contained in the block (except the assigned variable of an assignment). Show that the function $f_c : \mathcal{P}(\mathrm{Var})^B \to \mathcal{P}(\mathrm{Var})^B$ is $\sqcup$-continuous.

$$f_c(X)(b) = \left\{ x \,\middle|\, \exists b' \in B : b \to b' \text{ and } (b' \text{ uses } x \text{ or } (b' \neq [x := e]^\ell \text{ and } x \in X_{b'})) \right\}$$

Homework Exercise 2: Graph Reachability [8 points]
Find all vertices in the following graph $G = \langle V, E \rangle$, that are reachable from the initial nodes.



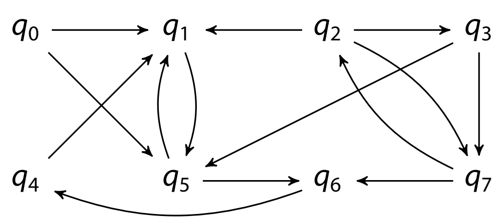
Let $v \in V$ be an initial vertex. Consider the function $f_v : \mathcal{P}(V) \to \mathcal{P}(V)$ with $f_v(X) := \{\, y \mid y = v \text{ or } \exists x \in X : \langle x, y \rangle \in E \,\}$.

a) [4 points] Compute $\mathrm{lfp}(f_{q_0})$ using the sequence in Kleene's fixed point theorem.

b) [4 points] Compute $\mathrm{lfp}(f_{q_2})$ using the sequence in Kleene's fixed point theorem.

Homework Exercise 3: Graph Unreachability [10 points]

Find all vertices in the following graph $G = \langle V, E\rangle$, that are **not** reachable from the start node q_0.



Consider the function $f : \mathcal{P}(V) \to \mathcal{P}(V)$ with $f(X) = \{ v \in V \mid v \neq q_0 \text{ and } (\forall x \in V \setminus X\colon \langle x, v\rangle \notin E) \}$.

a) [3 points] Show that f is monotone in $\langle \mathcal{P}(V), \subseteq\rangle$.

b) [3 points] Show that f is $\sqcap$-continuous in $\langle \mathcal{P}(V), \subseteq\rangle$.

c) [4 points] Compute gfp(f) using the sequence in Kleene's fixed point theorem.

Homework Exercise 4: Live Variables [9 points]

Consider the following program.

$[x := 0]^0$
while $[x^2 < y]^1$ **do**
    $[z := x + 1]^2$
    $[x := z^2 + z]^3$
end while
if $[x^2 = y]^4$ **then**
    $[z := 1]^5$
else
    $[z := y]^6$
end if
$[\mathrm{x} := \mathrm{z}]^7$

Map each block in the left program to the set of variables that may be read by some other block later in the program order.

a) [1 point] Draw the control flow graph G. Note that this is a backwards analysis.

b) [3 points] Consider the lattice $D = \langle \mathcal{P}(\{x, y, z\}), \subseteq\rangle$. Assign for each block $b \in B$ a suitable, monotone transfer function f_b over this lattice.

c) [5 points] Consider the data flow system $\langle G, D, \{x, y, z\}, (f_b)_{b\in B}\rangle$. Write down the induced equation system and determine its least solution using Kleene's fixed point theorem.

Exercise 5:

Consider the following program.

$[x := 0]^0$
while $[x < 2^4]^1$ **do**
  $[y := 3x + 2]^2$
  while $[y < 5x]^3$ **do**
    $[y := y + 2]^4$
    if $[3x < y]^5$ **then**
      $[x := x + 1]^6$
    end if
  end while
end while
$[x := x - 14]^7$

Map each block in the right program to the set of assignment blocks, that may have determined the current value of some variable when this block starts.

Draw the control flow graph G. Mark its extremal blocks. Note that this is a forwards analysis.

Consider the lattice $D = \langle \mathcal{P}(\{x, y\} \times (B + \{?\})), \subseteq \rangle$. Assign for each block $b \in B$ a suitable, monotone transfer function f_b over this lattice.

Consider the data flow system $\langle G, D, \{(x, ?), (y, ?)\}, (f_b)_{b \in B} \rangle$. Write down the induced equation system and determine its least solution using Kleene's fixed point theorem.