Expressiveness and analysis of valence automata over graph monoids

Georg Zetzsche

Technische Universität Kaiserslautern

Of stacks (of stacks (…) with blind counters) with blind counters

Georg Zetzsche

Technische Universität Kaiserslautern

Example (Pushdown automaton)

\[ a, \varepsilon, A \xrightarrow{\varepsilon, \varepsilon, \varepsilon} a, A, \varepsilon \]

\[ b, \varepsilon, B \xrightarrow{\varepsilon, \varepsilon, \varepsilon} b, B, \varepsilon \]
Example (Pushdown automaton)

\[ L = \{ w w^{\text{rev}} \mid w \in \{a, b\}^* \} \]
Example (Pushdown automaton)

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Example (Blind counter automaton)
Example (Pushdown automaton)

\[
\begin{align*}
q_0 & \xrightarrow{a, \varepsilon, A} q_0 \\
q_0 & \xrightarrow{b, \varepsilon, B} q_1 \\
q_1 & \xrightarrow{a, A, \varepsilon} q_1 \\
q_1 & \xrightarrow{b, B, \varepsilon} q_1
\end{align*}
\]

\[L = \{ww^{\text{rev}} \mid w \in \{a, b\}^*\}\]

Example (Blind counter automaton)

\[
\begin{align*}
q_0 & \xrightarrow{a, 1, 0} q_0 \\
q_0 & \xrightarrow{b, -1, -1} q_1 \\
q_1 & \xrightarrow{c, 0, 1} q_2
\end{align*}
\]

\[L = \{a^n b^n c^n \mid n \geq 0\}\]
Example (Partially blind counter automaton)

\[
\begin{array}{c}
q_0 \\
\downarrow a, 1 \\
\uparrow \\
\downarrow b, -1 \\
\end{array}
\quad
\begin{array}{c}
q_1 \\
\uparrow \varepsilon, 0 \\
\downarrow \varepsilon, -1 \\
\end{array}
\]

\[\text{for each prefix } p \text{ of } w, \quad |p(a, 1) + p(b, -1)| = |p| \]
Example (Partially blind counter automaton)

\[ L = \{ w \in \{ a, b \}^* \mid |p|_a \geq |p|_b \text{ for each prefix } p \text{ of } w \} \]
Automata models that extend finite automata by some storage mechanism:

- Pushdown automata
- Blind counter automata
- Partially blind counter automata
- Turing machines
Automata models that extend finite automata by some storage mechanism:

- Pushdown automata
- Blind counter automata
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- Turing machines

Each storage mechanism consists of:

- States: set $S$ of states
- Operations: partial maps $\alpha_1, \ldots, \alpha_n : S \rightarrow S$
<table>
<thead>
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<pre><code>                                |                           | $\text{pop}_a : wa \mapsto w, \ a \in \Gamma$ |
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| Partially blind counter      | $S = \mathbb{N}^n$ | $inc_i : (x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_i + 1, \ldots, x_n)$  
| automata                      |        | $dec_i : (x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_i - 1, \ldots, x_n)$  |

**Observation**

Here, a sequence $\beta_1, \ldots, \beta_k$ of operations is valid if and only if

$$\beta_1 \circ \cdots \circ \beta_k = \text{id}$$
Definition

A monoid is

- a set \( M \) together with
- an associative binary operation \( \cdot : M \times M \to M \) and
- a neutral element \( 1 \in M \) (\( 1a = a = a1 \) for any \( a \in M \)).
Definition

A monoid is

- a set $M$ together with
- an associative binary operation $\cdot : M \times M \to M$ and
- a neutral element $1 \in M$ ($a1 = 1a = a$ for any $a \in M$).

Storage mechanisms as monoids

- Let $S$ be a set of states and $\alpha_1, \ldots, \alpha_n : S \to S$ partial maps.
- The set of all compositions of $\alpha_1, \ldots, \alpha_n$ is a monoid $M$.
- The identity map is the neutral element of $M$.
- $M$ is a description of the storage mechanism.
Common generalization: Valence Automata

Valence automaton over $M$:

- Finite automaton with edges $p \xrightarrow{w|m} q$, $w \in \Sigma^*$, $m \in M$. 

Language class $\mathcal{V}A_p$ languages accepted by valence automata over $M$. 

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Valence automata

Common generalization: Valence Automata

Valence automaton over $M$:

- Finite automaton with edges $p \xrightarrow{w|m} q$, $w \in \Sigma^*$, $m \in M$.
- Run $q_0 \xrightarrow{w_1|m_1} q_1 \xrightarrow{w_2|m_2} \cdots \xrightarrow{w_n|m_n} q_n$ is accepting for $w_1 \cdots w_n$ if
  - $q_0$ is the initial state,
  - $q_n$ is a final state, and
Valence automata

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Valence automaton over $M$:

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  - $m_1 \cdots m_n = 1$. 

$\text{Language class } \text{VA}_p(M)$ languages accepted by valence automata over $M$. 

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Valence automata

Common generalization: Valence Automata

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Language class

$\text{VA}(M)$ languages accepted by valence automata over $M$. 

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Classical results can now be generalized:

Questions

- For which storage mechanisms can we avoid silent transitions?
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- For which do we have semilinearity of all languages?
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Classical results can now be generalized:

**Questions**

- For which storage mechanisms can we avoid silent transitions?
- For which do we have semilinearity of all languages?
- For which is the language class, for example, Boolean closed?
- For which can we decide, for example, emptiness?
Monoids defined by graphs

By graphs, we mean undirected graphs with loops allowed.

Let $\Gamma$ be a graph. Let $\tilde{\alpha}^t_{\tilde{\alpha}^t} \in X^\Gamma$.

Intuition: bicyclic monoid, $B = \alpha^t \tilde{\alpha}^t \in \varepsilon^u_i$.

$Z$: group of integers

For each unlooped vertex, we have a copy of $B$.

For each looped vertex, we have a copy of $Z$.

$M^\Gamma$ consists of sequences of such elements.

An edge between vertices means that elements can commute.
Monoids defined by graphs

By graphs, we mean undirected graphs with loops allowed. Let $\Gamma = (V, E)$ be a graph. Let

$$X_\Gamma = \{a_v, \overline{a}_v \mid v \in V\}$$
Monoids defined by graphs

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$$\cup \{xy = yx \mid x \in \{a_u, \bar{a}_u\}, y \in \{a_v, \bar{a}_v\}, \{u, v\} \in E\}$$
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$$\mathcal{M}_\Gamma = X_\Gamma^*/R_\Gamma$$
Monoids defined by graphs

By graphs, we mean undirected graphs with loops allowed. Let \( \Gamma = (V, E) \) be a graph. Let

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\[
\cup \{xy = yx \mid x \in \{a_u, \bar{a}_u\}, y \in \{a_v, \bar{a}_v\}, \{u, v\} \in E\}
\]

\[
M_\Gamma = X_\Gamma^*/R_\Gamma
\]

Intuition

- \( \mathbb{B} \): bicyclic monoid, \( \mathbb{B} = \{a, \bar{a}\}^*/\{a\bar{a} = \varepsilon\} \).
- \( \mathbb{Z} \): group of integers

For each unlooped vertex, we have a copy of \( \mathbb{B} \)

For each looped vertex, we have a copy of \( \mathbb{Z} \)

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Examples
Examples

\[ \mathbb{Z}^3 \]
Examples

$\mathbb{Z}^3$

Blind counter
Examples

Blind counter

$\mathbb{Z}^3$

$\cdots$

$\cdots$

$\cdots$
Examples

$$\mathbb{Z}^3$$

Blind counter

$$B \ast B \ast B$$
Examples

Blind counter

Pushdown
Examples

Blind counter

Pushdown

\[ \mathbb{Z}^3 \]

\[ B \cdot B \cdot B \]
Examples

Blind counter

Pushdown
Examples

Blind counter

Partially blind counter

Pushdown

Infinite tape (TM)
Examples

Blind counter

$\mathbb{Z}^3$

Pushdown

$\mathbb{B} \ast \mathbb{B} \ast \mathbb{B}$

Partially blind counter

$\mathbb{B}^3$
Examples

Blind counter

Pushdown

Partially blind counter
Examples

- Blind counter
- Pushdown

- Partially blind counter
- \( (B \ast B) \times (B \ast B) \)
Examples

\[ \mathbb{Z}^3 \]

Blind counter

\[ B \ast B \ast B \]

Pushdown

\[ B^3 \]

Partially blind counter

\[ (B \ast B) \times (B \ast B) \]

Infinite tape (TM)
Examples

Blind counter

\[ \mathbb{Z}^3 \]

Pushdown

\[ B \ast B \ast B \]

Partially blind counter

\[ B^3 \]

Infinite tape (TM)

\[ (B \ast B) \times (B \ast B) \]
Examples

Blind counter

Pushdown

Partially blind counter

Infinite tape (TM)
Examples

Blind counter

$\mathbb{Z}^3$

Pushdown

$B \ast B \ast B$

Partially blind counter

$B^3$

Infinite tape (TM)

$(B \ast B) \times (B \ast B)$
Examples

Blind counter

\[ \mathbb{Z}^3 \]

Pushdown

\[ \mathbb{B} \ast \mathbb{B} \ast \mathbb{B} \]

\[ (\mathbb{B} \ast \mathbb{B}) \times \mathbb{B} \times \mathbb{B} \]

Partially blind counter

\[ \mathbb{B}^3 \]

Infinite tape (TM)

\[ (\mathbb{B} \ast \mathbb{B}) \times (\mathbb{B} \ast \mathbb{B}) \]
Examples

Blind counter

\[ \mathbb{Z}^3 \]

Pushdown

\[ (B \times B) \times B \times B \]

Pushdown + partially blind counters

Partially blind counter

\[ B^3 \]

Infinite tape (TM)

\[ (B \times B) \times (B \times B) \]
Silent Transitions

A transition that reads no input is called *silent transition* or \( \varepsilon \)-transition.
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A transition that reads no input is called *silent transition* or $\varepsilon$-*transition*.

Important problem

- When can silent transitions be eliminated?
- Without silent transitions, membership in NP.
- Elimination can be regarded as a precomputation.
Silent Transitions

A transition that reads no input is called *silent transition* or *ε-transition*.

Important problem

- When can silent transitions be eliminated?
- Without silent transitions, membership in NP.
- Elimination can be regarded as a precomputation.

Question

For which storage mechanisms can we avoid silent transitions?
Theorem (Z., ICALP 2013)

Let $\Gamma$ be a graph such that

- any two looped vertices are adjacent,
- no two unlooped vertices are adjacent.

Silent transitions can be avoided over $\mathcal{M}_\Gamma$.

$\Gamma$ does not contain $\mathcal{P}$ as an induced subgraph.
Theorem (Z., ICALP 2013)

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Let $\Gamma$ be a graph such that

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Then the following conditions are equivalent:

- Silent transitions can be avoided over $\mathbb{M}\Gamma$.
- $\Gamma$ does not contain $\longrightarrow$ as an induced subgraph.
Theorem (Z., ICALP 2013)

Let $\Gamma$ be a graph such that

- any two looped vertices are adjacent,
- no two unlooped vertices are adjacent.

Then the following conditions are equivalent:

- Silent transitions can be avoided over $\overline{M}\Gamma$.
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Theorem (Z., ICALP 2013)

Let $\Gamma$ be a graph such that

- any two looped vertices are adjacent,
- no two unlooped vertices are adjacent.

Then the following conditions are equivalent:

- Silent transitions can be avoided over $M\Gamma$.
- $\Gamma$ does not contain $\xrightarrow{\cdot}$ as an induced subgraph.
- $M\Gamma \in \text{StCtr}$
Positive case

### Definition (Stacked counters)

Let StCtr be the smallest class of monoids such that

- $1 \in \text{StCtr}$
- if $M \in \text{StCtr}$, then $M \times \mathbb{Z} \in \text{StCtr}$
- if $M \in \text{StCtr}$, then $M \ast \mathbb{B} \in \text{StCtr}$
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**Interpretation of StCtr**

\( \text{StCtr} \) corresponds to the class of storage mechanisms obtained by

- Adding a blind counter \( (M \times \mathbb{Z}) \):
  - States: \((c, z)\), \( c \) an old state, \( z \in \mathbb{Z} \).
  - Operations: old operations; increment, decrement for counter
Positive case

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StCtr corresponds to the class of storage mechanisms obtained by

- adding a blind counter ($M \times \mathbb{Z}$):
  - States: $(c, z)$, $c$ an old state, $z \in \mathbb{Z}$.
  - Operations: old operations; increment, decrement for counter

- building stacks ($M \ast \mathbb{B}$)
  - States: sequences $c_1 \ldots c_n$, $c_i$ old states
  - Operations: push separator, pop if empty, manipulate topmost entry
Semilinearity

For which monoids $M$ are all languages in $\text{VA}(M)$ semilinear?

- Parikh’s Theorem: Pushdown automata
- Ibarra + Greibach: Blind counter automata

\[ \text{Let } \Gamma \text{ be a graph. The following conditions are equivalent:} \]

1. $\Gamma$ contains neither $\emptyset$ nor $\{1\}$ as an induced subgraph.
2. $\Gamma$, minus loops, is a transitive forest.
Semilinearity

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**Theorem (Buckheister, Z., MFCS 2013)**

Let $\Gamma$ be a graph. The following conditions are equivalent:

- All languages in $\text{VA}(\mathbb{M}\Gamma)$ are semilinear.
- $\Gamma$ satisfies:
  1. $\Gamma$ contains neither $\bullet \longrightarrow \bullet$ nor $\bullet \longrightarrow \bigcirc$ as an induced subgraph and
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- $\Gamma$ satisfies:
  1. $\Gamma$ contains neither $\bullet\longrightarrow\bullet$ nor $\bullet\longrightarrow\quad\longrightarrow\bullet$ as an induced subgraph and
  2. $\Gamma$, minus loops, is a transitive forest.
- $\text{VA}(M\Gamma) \subseteq \text{VA}(M)$ for some $M \in \text{StCtr}$. (NP-membership!)
Algebraic extensions

Let \( \mathcal{F} \) be a language class. An \( \mathcal{F} \)-grammar \( G \) consists of

- Nonterminals \( N \), terminals \( T \), start symbol \( S \in N \)
- Productions \( A \to L \) with \( L \subseteq (N \cup T)^* \), \( L \in \mathcal{F} \)
Expressiveness

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- Generated language: $\{w \in T^* \mid S \Rightarrow^* w\}$.
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- Such languages are *algebraic over* $\mathcal{F}$, class denoted $\text{Alg}(\mathcal{F})$. 
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Presburger constraints

For each language class $\mathcal{F}$, $\text{SLI}(\mathcal{F})$ denotes the class of languages

\[ h(L \cap \psi^{-1}(S)) \]

for some $L \in \mathcal{F}$, a homomorphism $h$ and a semilinear set $S$. 
A hierarchy of language classes

Hierarchy

\[ F_0 = \text{finite languages}, \]

\[ G_i = \text{Alg}(F_i), \quad F_{i+1} = \text{SLI}(G_i), \quad F = \bigcup_{i \geq 0} F_i. \]
A hierarchy of language classes

**Hierarchy**

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In particular: \( G_0 = \text{CF}. \)
A hierarchy of language classes

Hierarchy

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In particular: \( G_0 = \text{CF} \).

\[
F_0 \subseteq G_0 \subseteq F_1 \subseteq G_1 \subseteq \cdots \subseteq F
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In particular: \( G_0 = \text{CF}. \)

\[ F_0 \subseteq G_0 \subseteq F_1 \subseteq G_1 \subseteq \cdots \subseteq F \]

Theorem

\[ \text{VA}(B \ast B \ast M) = \text{Alg}(\text{VA}(M)) \]
A hierarchy of language classes

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In particular: \( G_0 = \text{CF} \).

\[ F_0 \subseteq G_0 \subseteq F_1 \subseteq G_1 \subseteq \cdots \subseteq F \]

Theorem

\[ \text{VA}(B \ast B \ast M) = \text{Alg}(\text{VA}(M)), \quad \bigcup_{i \geq 0} \text{VA}(M \times \mathbb{Z}^i) = \text{SLI}(\text{VA}(M)). \]
A hierarchy of language classes

**Hierarchy**

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In particular: \( G_0 = \text{CF} \).

\[ F_0 \subseteq G_0 \subseteq F_1 \subseteq G_1 \subseteq \cdots \subseteq F \]

**Theorem**

\[ \text{VA}(B \ast B \ast M) = \text{Alg}(\text{VA}(M)), \quad \bigcup_{i \geq 0} \text{VA}(M \times \mathbb{Z}^i) = \text{SLI}(\text{VA}(M)). \]

**Corollary**

*Stacked counter automata accept precisely the languages in \( F \).*
Downward closures

\[ u \preceq v: \text{ } u \text{ is obtained from } v \text{ by arbitrarily deleting symbols} \]
Downward closures

$u \preceq v$: $u$ is obtained from $v$ by arbitrarily deleting symbols

Theorem (Higman)

For every language $L \subseteq X^*$, the set $L_\downarrow = \{ u \in X^* \mid u \preceq v \text{ for some } v \in L \}$ is regular.
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**Applications**

- $L\downarrow$ is observed through a lossy channel.
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Computability

For which systems can we compute \( L_{\downarrow} \)?
Downward closures

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**Computability**

For which systems can we compute \( L\downarrow \)?

- for \( \text{Alg}(\mathcal{F}) \) whenever computable for \( \mathcal{F} \) (van Leeuwen 1978)
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**Computability**

For which systems can we compute \( L \downarrow \)?

- for Alg(\( F \)) whenever computable for \( F \) (van Leeuwen 1978)
- for Petri net languages (Habermehl, Meyer, Wimmel, ICALP 2010)
Computing the downward closure

**Theorem**

For stacked counter automata, downward closures can be computed.
Computing the downward closure

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**Problem**

- Computability preserved by \text{Alg}(\cdot)
Computing the downward closure

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- Computability preserved by Alg(·)
- Preservation not clear for SLI(·) (probably not true)
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For stacked counter automata, downward closures can be computed.

Problem

- Computability preserved by Alg(\cdot)
- Preservation not clear for SLI(\cdot) (probably not true)
- Hence: Stronger invariant

Parikh annotations

- New language in the same class
- Additional symbols encode decomposition of Parikh image into constant and period vectors
- Adding period vectors by inserting at designated positions
Parikh annotations

Example

\[ L = (ab)^*(ca^* \cup db^*) \]

Parikh image: \( (c + (a + b)^{\oplus} + a^{\oplus}) \cup (d + (a + b)^{\oplus} + b^{\oplus}) \).
Parikh annotations

Example

\[ L = (ab)^* (ca^* \cup db^*) \]

Parikh image: \((c + (a + b)^{+} + a^{+}) \cup (d + (a + b)^{+} + b^{+})\).

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\begin{align*}
P &= \{p, q, r, s\}, \\
C &= \{e, f\}, \\
P_e &= \{p, q\}, \\
P_f &= \{r, s\},
\end{align*}
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Parikh annotations

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\begin{align*}
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P_e &= \{p, q\}, & \varphi(p) &= a + b, & \varphi(q) &= a, \\
P_f &= \{r, s\}, & \varphi(r) &= a + b, & \varphi(s) &= b,
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- Makes Parikh decomposition accessible to transducers
Parikh annotations

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- Makes Parikh decomposition accessible to transducers
- Pumping lemma described by a language
Theorem

For each level $F_i$, one can compute Parikh annotations in $F_i$. 

Other applications of Parikh annotations include:

Theorem

For each $i \geq 0$: $F_i \sqsubseteq G_i \sqsubseteq F_i^{-1}$. 

Georg Zetzsche (TU KL)
Theorem

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Computing downward closures

Recursively with respect to the hierarchy level:

- For $G_i = \text{Alg}(F_i)$, use van Leeuwen’s algorithm
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Computing downward closures

Recursively with respect to the hierarchy level:

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- For $L \in F_i = \text{SLI}(G_{i-1})$
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*For each level $F_i$, one can compute Parikh annotations in $F_i$.***

Computing downward closures

Recursively with respect to the hierarchy level:

- For $G_i = \text{Alg}(F_i)$, use van Leeuwen’s algorithm
- For $L \in F_i = \text{SLI}(G_{i-1})$, write $L = h(L' \cap \psi^{-1}(S)), \ L' \in G_{i-1}$
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For each $i \geq 0$: $F_i \subsetneq G_i \subsetneq F_{i+1}$.
## Conclusion

- Silent transitions avoidable, non-uniform membership in NP

---

Parikh's Theorem holds

Downward closure computable

Strict hierarchy of language classes

More classical results can be generalized:

Ongoing work

- Uniform word problem, connections to group theory
- Decidability of logics over reachability graphs
- $\preceq$/$\succeq$ computation
- Decidability of questions for Büchi variants
Conclusion

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